JOURNAL OF
SYMPLECTIC GEOMETRY
Volume 4, Number 3, 237–257, 2007

ZERO-SETS OF NEAR-SYMPLECTIC FORMS

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We give elementary proofs of two 'folklore' assertions about near-symplectic forms on four-manifolds: that any such form can be modified, by an evolutionary process taking place inside a finite set of balls, so as to have any chosen positive number of zero-circles; and that, on a closed manifold, the number of zero-circles for which the splitting of the normal bundle is trivial has the same parity as $1 + b_1 + b_2^+$.

1. Statement of results

In this paper, we establish some properties of near-symplectic forms, as defined in [**1**]:

Definition 1.1. A two-form ω on an oriented four-manifold X is called *near-positive* if at each point $x \in X$, either (i) $(\omega \wedge \omega)(x) > 0$, or (ii) $\omega(x) = 0$ and the intrinsic gradient $(\nabla\omega)(x)\colon T_xX \to \Lambda^2 T_x^*X$ has rank 3 as a linear map. It is *near-symplectic* if, in addition, $d\omega = 0$.

To clarify: at a point where $\omega(x) = 0$, there is an intrinsic gradient $(\nabla_v\omega)(x) \in \Lambda^2 T_x^*X$ in the direction v, for any $v \in T_xX$, because x is a zero of a smooth section of a vector bundle. If $\omega \wedge \omega \geq 0$ in a neighbourhood of x then $(\nabla_v\omega) \wedge (\nabla_v\omega) \geq 0$ for all $v \in T_xX$. The wedge-square quadratic form $\Lambda^2 T_x^*X \otimes \Lambda^2 T_x^*X \to \Lambda^4 T_x^*X \to \mathbb{R}^4$, which is defined up to a positive scalar, has signature $(3, 3)$, so to say that $(\nabla\omega)(x)$ has rank 3 is to say that its image is a maximal positive-definite subspace for the wedge-square form.

Lemma 1.2. *The zero-set $Z_\omega = \{x \in X : \omega_x = 0\} \subset X$ of a near-positive form $\omega \in \Omega^2_X$ is a smooth* 1*-dimensional submanifold.*

Proof. Take $z \in Z_\omega$. Working over a small ball $B \ni z$, choose a three-plane subbundle $E \subset \Lambda^2 T^*B$ such that E_z is complementary to $\mathrm{im}(\nabla\omega)(z)$. Project $\omega|_B$ to a section $\overline{\omega}$ of $(\Lambda^2 T^*B)/E$. Then $(\nabla\overline{\omega})(z)$ is surjective, so, shrinking B if necessary, $\overline{\omega}$ vanishes transversely along a 1-submanifold