On the Finite Difference Approximation for a Parabolic Blow-Up Problem

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1. Introduction

We consider a scalar semi-linear parabolic partial differential equation $u_t = u_{xx} + f(u)$ ($0 \le t$, $0 \le x \le 1$) with the Dirichlet boundary condition $u = 0$ on $x = 0$ and $x = 1$, where $f\colon \mathbb{R} \to \mathbb{R}$ is a smooth function. It is known (see, [4, 8, 10, 11]) that a solution with a large initial data blows up in finite time, if an appropriate growth condition on f as $u \to \infty$ is imposed. We are concerned in this paper with a question as to how a finite difference scheme can reproduce the blow-up phenomena. By studying various papers, we found that many interesting problems for numerical analysis of the parabolic blow-up problem are left unsolved. And we would like to solve some of them in the present paper.

Let us recall a pioneering paper by Nakagawa [12]. With $f(u) = u^2$, Nakagawa [12] considered a finite difference scheme, where a uniform mesh was used for the spatial variable and a certain adaptive mesh was used for the time variable. He then showed that his finite difference solutions blow up in finite time if a certain largeness condition on the initial data is assumed and that the numerical blow-up time converges to the 'real' blow-up time if the mesh size tends to zero. Let $T > 0$ be the blow-up time. Then it is not difficult to prove that the finite difference scheme converges in $0 \le t \le \tilde{T}$ for any prescribed $\tilde{T} < T$. This is a classical result, but Nakagawa did much more. His result is important in that he proved the convergence up to the blow-up time. Later, a finite element analogue was considered by Nakagawa and Ushijima [13]. Quite general blow-up conditions were established for semi-discretized equations by T.K. Ushijima [16]. See also [1, 2]. Recently a substantial generalization was made on the convergence of the blow-up time by Abia and others [3]. A different approach was proposed by Hirota and Ozawa [9].

A new direction was explored by Chen [6], who considered the equation with $f(u) = u^{1+\alpha}$ and proved, among others, that some numerical solutions can blow up at more than one point, while a one-point blow-up is known to occur in the continuous problem. Thus, a finite difference scheme with a spatially uniform mesh does not correctly reproduce the blow-up phenomena. Correct meaning will be stated later.