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BOOK REVIEW

A Spinorial Approach to Riemannian and Conformal Geometry, by Jean-Pierre Bourguignon, Oussama Hijazi, Jean-Louis Milhorat, Andrei Moroianu and Sergiu Moroianu, European Mathematical Society, Switzerland 2015, ix+452 pp, ISBN 978-3-03719-136-1

The Dirac equation (in natural units $c = \hbar = 1$)

$$\mathrm{i}\sum_{\nu=0}^{3}\gamma^{\mu}\partial_{\mu}\psi - m\psi = 0$$

was formulated by P.A.M. Dirac in 1928. It is one of the most fundamental equations of quantum mechanics, to be explicit in this equation $\partial_\mu = \frac{\partial}{\partial x^\mu}$, $\psi(x) = \psi(x^0, \ldots, x^3) \in \mathbb{C}^4, x \in \mathbb{R}^{1,3}$, is a complex four-vector valued function on $\mathbb{R}^{1,3}[x^0, x^1, x^2, x^3]$, called a spinor field, and γ^μ are 4×4 matrices, the so-called Dirac's γ-matrices. They satisfy $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$, where $\eta^{\mu\nu}$ is the Minkowski tensor. The Dirac equation is invariant with respect to translations and a specific action of the connected double cover of the Lorentz group $\mathrm{O}(1,3)$ called the **spinor representation**. Applying the operator $\mathrm{i}\sum_{i=0}^{3}\gamma^\mu\partial_\mu + m$ to the Dirac equation, one obtains the so-called Klein–Gordon equation $\Box\psi + m^2\psi = 0$, where $\Box = \partial^2_{x^0} - \partial^2_{x^1} - \partial^2_{x^2} - \partial^2_{x^3}$ is the wave operator. Let us note that a kind of spinors was already known to Pauli who postulated his equation similar to the Dirac equation in the realm of non-relativistic quantum mechanics. Although its relevance is rather limited in quantum theory, the Dirac equation is still a helpful tool in physics and a source of inspiration for analysts and geometers. In mathematics, one usually considers Riemannian manifolds rather than the Lorentzian ones. The appropriate Dirac operator is adapted to the Riemannian metric. Its square is an elliptic operator of Laplace type. In the reviewed book, the authors consider Riemannian manifolds and appropriate elliptic Dirac operators. Spinors and Dirac operators are also used in non-commutative geometry of A. Connes, serve as an important example for applications of the Atiyah–Singer index theorem, and a spectrum of the Dirac operator is helpful in determining of topological properties of the underlying manifold.