Homology, Homotopy and Applications, vol. 8(2), 2006, pp.157–203

A CONSTRUCTION OF QUOTIENT A_∞-CATEGORIES

VOLODYMYR LYUBASHENKO AND SERGIY OVSIENKO

(*communicated by Jim Stasheff*)

Abstract

We construct an A_∞-category $\mathsf{D}(\mathcal{C}|\mathcal{B})$ from a given A_∞-category $\mathcal{C}$ and its full subcategory $\mathcal{B}$. The construction is similar to a particular case of Drinfeld's construction of the quotient of differential graded categories. We use $\mathsf{D}(\mathcal{C}|\mathcal{B})$ to construct an A_∞-functor of K-injective resolutions of a complex, when the ground ring is a field. The conventional derived category is obtained as the 0-th cohomology of the quotient of the differential graded category of complexes over acyclic complexes. This result follows also from Drinfeld's theory of quotients of differential graded categories.

Introduction

In [**Dri04**] Drinfeld reviews and develops Keller's construction of the quotient of differential graded categories [**Kel99**] and gives a new construction of the quotient. This construction consists of two parts. The first part replaces the given pair $\mathcal{B} \subset \mathcal{C}$ of a differential graded category $\mathcal{C}$ and its full subcategory $\mathcal{B}$ with another such pair $\widetilde{\mathcal{B}} \subset \widetilde{\mathcal{C}}$, where $\widetilde{\mathcal{C}}$ is homotopically flat over the ground ring $\Bbbk$ (K-flat) [**Dri04**, Section 3.3], and there is a quasi-equivalence $\widetilde{\mathcal{C}} \to \mathcal{C}$ [**Dri04**, Section 2.3]. The first step is not needed, when $\mathcal{C}$ is already homotopically flat, for instance, when $\Bbbk$ is a field. In the second part, a new differential graded category $\mathcal{C}/\mathcal{B}$ is produced from a given pair $\mathcal{B} \subset \mathcal{C}$, by adding to $\mathcal{C}$ new morphisms $\varepsilon_U : U \to U$ of degree -1 for every object U of $\mathcal{B}$, such that $d(\varepsilon_U) = \mathrm{id}_U$.

In the present article we study an A_∞-analogue of the second part of Drinfeld's construction. Namely, to a given pair $\mathcal{B} \subset \mathcal{C}$ of an A_∞-category $\mathcal{C}$ and its full subcategory $\mathcal{B}$, we associate another A_∞-category $\mathsf{D}(\mathcal{C}|\mathcal{B})$ via a construction related to the bar resolution of $\mathcal{C}$. The A_∞-category $\mathsf{D}(\mathcal{C}|\mathcal{B})$ plays the role of the quotient of $\mathcal{C}$ over $\mathcal{B}$ in some cases, for instance, when $\Bbbk$ is a field. When $\mathcal{C}$ is a differential graded category, $\mathsf{D}(\mathcal{C}|\mathcal{B})$ is precisely the category $\mathcal{C}/\mathcal{B}$ constructed by Drinfeld [**Dri04**, Section 3.1].

The research of V. L. was supported in part by grant 01.07/132 of State Fund for Fundamental Research of Ukraine.
Received April 13, 2003, revised October 30, 2003, August 7, 2006; published September 30, 2006.
2000 Mathematics Subject Classification: 16E45, 18G10, 18G55, 57T30.
Key words and phrases: A_∞-category, A_∞-functor, A_∞-transformation, K-injective resolution, quotient A_∞-category.