The Number of Real Quadratic Fields Having Units of Negative Norm

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CONTENTS

We study the density of the set of real quadratic fields for which the norm of the fundamental unit equals -1 inside the set of real quadratic fields containing elements of norm -1. A conjectural density is derived from a single heuristic assumption, and experimental data supporting this assumption are given. We finally discuss how close one can get to proving such conjectural densities.

1. INTRODUCTION

The main problem in this paper, although formulated and treated in terms of real quadratic number fields, is a very old problem that does not need anything in its formulation beyond ordinary integers. More precisely, we will be concerned with the solvability of the *negative* Pell equation

$$x^2 - dy^2 = -1 \quad \text{with } x, y \in \mathbf{Z} \tag{1.1}$$

for squarefree numbers $d \in \mathbf{Z}_{>1}$. The solvability of this equation was studied by Euler, who mistakenly attached the name of the English mathematician John Pell (1611–1685) to the related equation $x^2 - dy^2 = 1$. The problem of finding nontrivial solutions to the Pell equation itself had already been studied by many mathematicians long before Euler's time, and Fermat, who posed it as a challenge to the English mathematicians in 1657, knew that it was solvable for all nonsquare $d > 1$. An excellent account of the long history of the equation can be found in [Weil 1984].

In contrast to the relatively straightforward answer in the case of the Pell equation itself, the solvability of the negative Pell equation we are dealing with here turns out to be a much more complicated

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