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A USEFUL CHARACTERIZATION OF CLARKE DERIVATIVES

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Abstract. A characterization of Clarke's directional derivatives as the upper semicontinuous regularization of Dini's directional derivatives is proved. We extend some well known results in Nonsmooth Analysis by using this characterization, which, moreover, allows a great simplification in the proofs.

1. Introduction. In 1973, Frank H. Clarke introduced a generalized notion of differentiability well suited to the analysis of non (necessarily) differentiable locally Lipschitzian real valued functions defined on $\mathbf{R}^n$. The starting point of his 1973 definition was a theorem by Rademacher which asserts that these functions are differentiable almost everywhere. This fact permitted Clarke to define the generalized gradient of a locally Lipschitzian function $f : \mathbf{R}^n \to \mathbf{R}$ at a point $a \in \mathbf{R}^n$ as the compact and convex set

$$\partial f(a) = \text{co}\{\lim \nabla f(x_i)/x_i(\in \text{ dom}\,\nabla f) \to a\} \tag{1.1}$$

and characterize the support functional to this set

$$f^\circ(a;v) = \limsup_{x\in(\text{dom}\,\nabla f)\to a} \langle \nabla f(x), v\rangle \tag{1.2}$$

as

$$f^\circ(a;v) = \limsup_{\substack{x\to a\\ t\downarrow 0}} t^{-1}[f(x+tv) - f(x)] \tag{1.3}$$

which is known as the Clarke generalized directional derivative of f at a in the direction $v \in \mathbf{R}^n$.

When $f'(x;v) = \lim_{t\downarrow 0} t^{-1}[f(x+tv) - f(x)]$ exists for all x, $v \in \mathbf{R}^n$, from (1.2) we obtain the characterization

$$f^\circ(a;v) = \limsup_{x\to a} f'(x;v) \tag{1.4}$$

as was noticed by Hiriart-Urruty [11] and Rockafellar [15]. Formula (1.4) is often more convenient than (1.3) and (1.2), but has a big disadvantage, since the function f must be assumed directionally differentiable everywhere.

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