

# A GENERAL EXISTENCE AND UNIQUENESS THEORY FOR IMPLICIT DIFFERENTIAL-ALGEBRAIC EQUATIONS\*

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**Abstract.** This paper presents a general existence and uniqueness theory for differential-algebraic equations extending the well-known ODE theory. Both local and global aspects are considered, and the definition of the index for nonlinear problems is elucidated. For the case of linear problems with constant coefficients the results are shown to provide an alternate treatment equivalent to the standard approach in terms of matrix pencils. Also, it is proved that general differential-algebraic equations carry a geometric content, in that they are locally equivalent to ODEs on a “constraint” manifold. A simple example from particle dynamics is given to illustrate our approach.

**1. Introduction.** Differential-algebraic equations (DAEs) are frequently identified as implicit equations

$$F(t, x, x') = 0$$

for which the derivative  $x'$  cannot be expressed explicitly as a function of  $t$  and  $x$  (see e.g., [1]). In particular, if  $x \in \mathbb{R}^n$  and  $F$  maps into  $\mathbb{R}^n$ , this includes the case when the partial derivative  $D_p F(t, x, p)$  of  $F$  with respect to its third variable  $p$  is not surjective. More specifically, in the setting of DAEs it is natural to require the stronger hypothesis that  $D_p F(t, x, p)$  has constant rank on the domain under consideration. Indeed, the prototype for such equations is given by

$$F(t, x, p) = \begin{pmatrix} F_1(t, x) \\ F_2(t, x, p) \end{pmatrix},$$

where  $F_1$  and  $F_2$  map into  $\mathbb{R}^{n-r}$  and  $\mathbb{R}^r$ , respectively, and  $D_p F_2(t, x, p)$  has full rank, so that, indeed,  $D_p F(t, x, p)$  has constant rank  $r < n$ .

Many DAE-problems of practical interest do not exhibit such a convenient splitting between algebraic and differential parts as in (1.2). Moreover, even if the equations can be written in the separated form (1.2), the rank of  $D_p F_2(t, x, p)$  may turn out to be less than  $r$  so that  $F_2(t, x, x') = 0$  is an equation containing an implicit algebraic part. While these comments suggest the need for a thorough investigation of DAEs in the broad setting of (1.1), existence and uniqueness theories

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