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# QUADRATIC SYSTEMS WITH A CRITICAL POINT OF HIGHER MULTIPLICITY

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**Abstract.** We study plane quadratic systems which have a critical point of multiplicity greater than two, either in the finite plane or at infinity, and show that in many cases they have at most one limit cycle.

**1. Introduction.** Let

$$x' = P(x,y), \quad y' = Q(x,y) \tag{1}$$

be a holomorphic system of differential equations with an isolated critical point $M = (x_0, y_0)$. We define the *multiplicity* of the critical point $M$ to be its multiplicity as a point of intersection of the analytic curves $P = 0$ and $Q = 0$; i.e., it is the greatest integer $m$ for which there exist functions $f_1(x,y), \ldots, f_m(x,y)$, holomorphic at $M$ and such that if

$$c_1 f_1 + \cdots + c_m f_m = \alpha P + \beta Q,$$

where $c_1, \ldots, c_m$ are complex numbers and $\alpha(x,y)$, $\beta(x,y)$ are functions holomorphic at $M$, then $c_1 = \cdots = c_m = 0$. By taking $f_1 = 1$ we see that $m \geq 1$. By taking the Jacobian matrix of $P$ and $Q$ to be in triangular form at $M$ we see that the critical point $M$ has multiplicity 1 if and only if it is *elementary*; i.e., the Jacobian determinant $P_x Q_y - P_y Q_x$ is nonzero at $M$.

Suppose the critical point $M$ is *semi-elementary*; i.e., the Jacobian determinant vanishes at $M$ but the divergence $P_x + Q_y$ does not. Without loss of generality we may assume $M = (0,0)$ and

$$P(x,y) = a_{20}x^2 + a_{11}xy + a_{02}y^2 + \ldots$$
$$Q(x,y) = y + b_{20}x^2 + b_{11}xy + b_{02}y^2 + \ldots.$$

Then near the origin the curve $Q(x,y) = 0$ has the form $y = \varphi(x)$, where $\varphi$ is holomorphic and $\varphi(0) = \varphi'(0) = 0$. Since the origin is assumed to be an isolated critical point we have

$$\psi(x) := P[x, \varphi(x)] = \Delta_m x^m + \ldots,$$

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