Differential and Integral Equations, Volume 3, Number 4, July 1990, pp. 639–642.

MULTIVALUED DIFFERENTIAL EQUATIONS ON CLOSED SETS II

KLAUS DEIMLING
FB 17 der Universität, Warburger Str. 100, D-4790 Paderborn (FRG)

Abstract. Let $X = \mathbb{R}^n$, $D \subset X$ closed, $J = [0, a] \subset \mathbb{R}$ and $F : J \times D \to 2^X \backslash \emptyset$ be a multivalued map. We consider the initial value problem

$$u' \in F(t, u) \quad \text{a.e. on } J, \;\; u(0) = x_0 \in D, \tag{1}$$

where solutions are understood to be absolutely continuous (ac) on J, and prove two existence theorems for the cases of main interest, namely the upper semicontinuous (usc) one under condition $F(t,x) \cap T_D(x) \neq \emptyset$ on $J \times D$ and the almost lower semicontinuous (lsc) one under condition $F(t,x) \subset T_D(x)$ on $J \times D$, where, for $x \in D$,

$$T_D(x) = \{ y \in X : \underline{\lim}_{\lambda \to 0^+} \lambda^{-1} \rho(x + \lambda y, D) = 0 \} \quad \text{with} \quad \rho(x, D) = \inf_D |x - y|. \tag{2}$$

1. The usc case. This paper is a continuation of [3]. We keep the notations and definitions given there, but consider only $X = \mathbb{R}^n$. Possible extensions to Banach spaces X with $\dim X = \infty$ will be indicated in [5].

Theorem 1. *Let $X = \mathbb{R}^n$, $D \subset X$ be closed, $J = [0, a] \subset \mathbb{R}$ and $F : J \times D \to 2^X \backslash \emptyset$ be such that*

(i) *$F(t,x)$ is closed convex for all $(t,x) \in J \times D$,*

(ii) *$\|F(t,x)\| \leq c(t)(1 + |x|)$ on $J \times D$ with $c \in L^1(J)$,*

(iii) *$F(\cdot, x)$ is measurable, $F(t, \cdot)$ is usc,*

(iv) *$F(t,x) \cap T_D(x) \neq \emptyset$ on $J \times D$*

are satisfied. Then (1) has a solution.

Proof: Using Theorem 2 in [7] or the paper of Jarnik/Kurzweil mentioned there, we find an $F_0 : J \times D \to 2^X$ with compact convex $F_0(t,x) \subset F(t,x)$ such that

(a) if u, v are measurable and $v(t) \in F(t, u(t))$ on J, then $v(t) \in F_0(t, u(t))$ almost everywhere on J and

(b) to $\epsilon > 0$ there exists a closed $J_\epsilon \subset J$ with Lebesgue measure $\mu(J \backslash J_\epsilon) \leq \epsilon$ such that $F_0(t,x) \neq \emptyset$ on $J_\epsilon \times D$ and $F_0\big|_{J_\epsilon \times D}$ is usc.

Received for publication August 22, 1989.
AMS Subject Classifications: 34A60, 34G20, 47H10.