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SINGULAR SELF-ADJOINT STURM-LIOUVILLE PROBLEMS

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Abstract. Singular self-adjoint boundary conditions for Sturm-Liouville (S-L) problems are characterized. We believe this characterization is (i) new, (ii) simpler and more explicit than the well known characterization, and (iii) it is an exact parallel of the regular case.

1. Introduction. There are two fundamental classes of boundary value problems for the Sturm-Liouville (S-L) expression

$$My = \frac{1}{w}[-(py')' + qy] \quad \text{on} \quad I = (a, b), \quad -\infty \leq a < b \leq \infty; \tag{1.1}$$

i.e., regular and singular. In both cases, the boundary conditions required to obtain self-adjoint realizations of M are well known (and have been known for over a century). For details, see the book by Naimark [1968]. In the regular case, these conditions can be interpreted as linear combinations of the values of the function y and its quasi-derivative py' at the end points a and b. Such a representation is not possible at a singular end point c, say, because $y(c)$ and $(py')(c)$ do not exist even in a limiting sense, in general. The known characterization of the singular self-adjoint boundary conditions involves the sesquilinear form associated with M and elements of the maximal domain. In this paper, we show that the characterization of the singular self-adjoint boundary conditions is identical to that in the regular case provided that y and py' are replaced by certain Wronskians involving y and two linearly independent solutions of $My = 0$.

Notation and basic assumptions. The real valued Lebesgue measurable functions p, q and w are assumed to satisfy the following basic conditions:

$$p^{-1},\, q,\, w \in L_{\text{loc}}(I), \quad w(t) > 0 \text{ a.e.} \tag{1.2}$$

These conditions are assumed to hold throughout this paper. The local integrability conditions of (1.2) are necessary and sufficient for arbitrary initial value problems, at any point c in I of the equation $My = \lambda wy$, $\lambda \in C$, to have unique solutions.

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