Differential and Integral Equations, Volume 3, Number 1, January 1990, pp. 103–125.

# REGULARITY RESULTS FOR FIRST ORDER HAMILTON–JACOBI EQUATIONS

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**Introduction.** We are interested here in regularity results for the first-order Hamilton-Jacobi equations

$$H(x, y, u, D_x u, D_y u) = 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R}^m, \tag{1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $H$ is a given continuous function and $u$ is a real-valued function that we will always assume to be bounded uniformly continuous (BUC in short). Our work relies on the notion of viscosity solutions, introduced by M.G. Crandall and P.L. Lions [10] (see also M.G. Crandall, L.C. Evans and P.L. Lions [8] and P.L. Lions [24]) and in all the following, all the equations or inequalities have to be understood in viscosity sense. We refer the reader to the bibliography for references concerning the study of the properties of viscosity solutions (existence, uniqueness, stability, ... ) but our list is by no means complete. Our aim is to give general sufficient conditions on $H$, ensuring a Hölder modulus of continuity for $u$ *in $x$ only.* We will also consider the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + H(x, y, u, D_x u, D_y u) = 0 & \text{in } \mathbb{R}^n \times \mathbb{R}^m \times (0, T), \\ u(x, y, 0) = u_0(x, y) & \text{in } \mathbb{R}^n \times \mathbb{R}^m, \end{cases} \tag{2}$$

where we give more particular results concerning the growth of $u$ in $t$. Roughly speaking, we were able to estimate separately $(u_t)^+$ and $(u_t)^-$ because of the particular structure of the equation (linear dependence in $u_t$).

Results of the same type as ours are given in P.L. Lions [25], [26], M.G. Crandall, P.L. Lions and P.E. Souganidis [13] and G. Barles [2]. In [25], existence results for (1) and (2) of Hölder continuous solutions (in $x$ and $y$) are given; [13] and [26] consider regularizing effects for (2). As in [2], we will have these two types of results.

The spirit of this work is almost the same as the one of [2]: in order to give the most general results, we are going to use three main ideas: the first one is that, a priori, we must only need assumptions on $H(x, y, u, p_1, p_2)$ for $|p_1|$ large. Indeed, since $p_1$ stands for $D_x u$, if $p_1$ is bounded, intuitively, we are done. The second

Received December 28, 1988.
AMS Subject Classifications: 35D10, 35F25, 35F30.