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OMEGA LIMIT SETS OF NONEXPANSIVE MAPS: FINITENESS AND CARDINALITY ESTIMATES

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Introduction. If (M, d) is a complete metric space, C is a closed subset of M and $T : C \to C$ a map, then for $x \in C$ one can define the omega limit set $\omega(x; T)$:

$$\omega(x;T) = \omega(x) = \cap_{k\geq 1}\mathrm{cl}(\cup_{j\geq k}T^j(x)). \tag{1}$$

In equation (1), $\mathrm{cl}(A)$ denotes the closure of a set A. Alternatively, $\omega(x;T)$ is the set of $y \in M$ such that there exists a sequence of integers $k_i \to \infty$ such that

$$y = \lim_{i\to\infty} T^{k_i}(x).$$

It is well-known that $\omega(x)$ is closed, $T(\omega(x)) \subset \omega(x)$ and $T(\omega(x)) = \omega(x)$ if T is continuous and C is compact. The map T is called nonexpansive (with respect to d) if

$$d(Tx, Ty) \leq d(x, y) \quad \text{for all } x, y \in C. \tag{2}$$

If $\omega = \omega(x;T)$ is nonempty and T is nonexpansive it is known (see [5]) that $\omega(y;T) = \omega(x;T)$ for all $y \in \omega(x;T)$ and that the restriction of T to $\omega(x;T)$ is an isometry of $\omega(x;T)$ onto itself. In particular, for each $y, z \in \omega(x;T)$ there exists a sequence of integers $k_i \to \infty$ such that

$$\lim_{i\to\infty} T^{k_i}(y) = z. \tag{3}$$

If $\omega(x;T)$ is also compact, then by using the Ascoli-Arzela theorem and equation (3), one easily shows (see Lemma 1 in [7]) that for each y and z in $\omega(x;T)$ there exists an isometry $S_{y,z} : \omega(x;T) \to \omega(x;T)$ of $\omega(x;T)$ onto itself such that $S_{y,z}(y) = z$ and such that

$$S_{y,z}S_{u,v} = S_{u,v}S_{y,z} \quad \text{for all } y, z, u, v \in \omega(x;T). \tag{4}$$

If $M = \mathbb{R}^n$, C is compact, the metric d arises from the ℓ_1-norm

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|, \tag{5}$$

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