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MINIMAL SOLUTIONS OF MULTIVALUED DIFFERENTIAL EQUATIONS

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Abstract. Let $J = [0, a] \subset \mathbb{R}$, $2^X \backslash \emptyset$ be the nonempty subsets of $X = \mathbb{R}^n$ and $F : J \times X \to 2^X \backslash \emptyset$ a multivalued map. We consider the initial value problem

$$u' \in F(t, u) \quad \text{a.e. on } J,\ u(0) = x_0 \tag{1}$$

and prove the existence of a minimal solution u_* of (1), where a solution of (1) is understood to be an a.c. (absolutely continuous) function $u : J \to X$ and "minimal" refers to the partial ordering induced by a cone $K \subset X$. This problem was solved in [2] for every cone with nonempty interior $\overset{\circ}{K}$ in case $F(t, \cdot)$ is continuous. If the $F(t, \cdot)$ are only upper semicontinuous the problem is harder and existence of u_* was proven in [3] for special cones. Now we are able to get u_* for every cone K with $\overset{\circ}{K} \neq \emptyset$ in case F is almost usc.

1. Preliminaries. (i) Let us first recall some definitions. Given metric spaces X and Y, an $F : Y \to 2^X \backslash \emptyset$ is ϵ-δ-usc [ϵ-δ-lsc] if to every $y_0 \in Y$ and $\epsilon > 0$ there is $\delta = \delta(y_0, \epsilon) > 0$ such that

$$F(y) \subset F(y_0) + B_\epsilon(0) \qquad [F(y_0) \subset F(y) + B_\epsilon(0)] \quad \forall y \in B_\delta(y_0).$$

If the $F(y)$ are compact, this coincides with the usual definition of usc (upper semicontinuous) and lsc (lower semicontinuous), respectively. In case $Y = J \times X$ we call F almost usc if to $\epsilon > 0$ there exists a closed $J_\epsilon \subset J$ with $\mu(J \backslash J_\epsilon) \leq \epsilon$ such that $F\big|_{J_\epsilon \times X}$ is usc. $G : J \to 2^X \backslash \emptyset$ will be called measurable if $G^{-1}(O)$ is Lebesgue measurable for every open $O \subset X$.

By a cone we mean a closed convex $K \subset X$ satisfying $\lambda K \subset K$ for all $\lambda \geq 0$ and $K \cap (-K) = \{0\}$. With $K^* = \{x^* \in X^* : x^*(x) \geq 0 \text{ on } K\}$ we let $f : J \times X \to X$ be *quasimonotone* with respect to K if $x^*\big(f(t, x + y) - f(t, x)\big) \geq 0$ whenever $t \in J$, $x \in X$, $y \in K$, $x^* \in K^*$ and $x^*(y) = 0$. We also use the fact that every cone $K \subset \mathbb{R}^n$ with $\overset{\circ}{K} \neq \emptyset$ is fully regular; i.e., every bounded increasing sequence is convergent; see, e.g., §19 in [1].

(ii) To get the minimal solution u_* of (1), the considerations in [2], [3] show that it is reasonable to assume

$$F(t, x) \text{ is compact convex, for every } (t, x) \in J \times X \tag{2}$$

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