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INTERIOR GRADIENT BOUNDS FOR THE MEAN CURVATURE EQUATION BY VISCOSITY SOLUTIONS METHODS

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Introduction. We give a new proof of the existence of an interior gradient bound for the prescribed Mean Curvature Equation

$$\begin{aligned} Mu &= -(1+|Du|^2)\Delta u + D_{ij}uD_iuD_ju \\ &= H(x)(1+|Du|^2)^{3/2} \quad \text{in } \Omega, \end{aligned} \tag{0.1}$$

where Ω is a domain in $\mathbb{R}^n$ and H is a Lipschitz continuous function on $\bar{\Omega}$.

We refer to Gilbarg and Trudinger [10] for a complete presentation of this equation and of its historical background. We just briefly recall that the first results concerning interior gradient bounds were obtained by Finn [9] in the case when $n = 2$ and for the minimal surface equation—the case $H \equiv 0$. Then, Bombieri, De Giorgi and Miranda [4] extended this result to the case of any dimension. Interior gradient bounds for the general Mean Curvature Equation were first proved by Ladyzhenskaya and Ural'tseva [18]. Later the proofs of the above papers were simplified, refined and applied to more general equations by Michael and Simon [22], Simon [24], and Trudinger [25–26]. All these works use variational methods and a geometrical approach by considering the hypersurface in $\mathbb{R}^{n+1}$

$$\sum = \{(x, x_{n+1}) \in \Omega \times \mathbb{R} / x_{n+1} = u(x)\}.$$

The key point of their proofs is an isoperimetric inequality of Federer and Fleming (see [8]) and its resulting Sobolev inequality. To our knowledge, this type of method was, so far, the only known for proving interior gradient bounds.

Our approach is completely different. It relies on the notion of viscosity solutions; this notion of weak solutions was first introduced for first-order fully nonlinear equations by Crandall and Lions [6] (see also Crandall, Evans, and Lions [5]). Its extension to second-order fully nonlinear elliptic equations, first given in Lions [19], faced a long time the lack of uniqueness results. But Jensen [13] first breaks this difficulty by using the Alexandrov Maximum Principle and recently Jensen [14], Ishii [11] and Ishii and Lions [12] proved independently very general uniqueness

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