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ON THE INVERSION OF THE LAGRANGE-DIRICHLET THEOREM IN A CASE OF NONHOMOGENEOUS POTENTIAL

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Abstract. We consider a natural Lagrangian system. We assume that the potential $U(q)$ has a critical point $q = 0$, which is not a local maximum; furthermore this property depends only on a suitable k-jet of U ($k \geq 3$). Then, if the Lagrangian function is $\mathcal{C}^h$, $h \geq k+3$, and if a "weak coupling" condition is satisfied, we prove that $q = 0$ is an unstable equilibrium position. The last condition can be removed if the regularity of the Lagrangian function is strengthened by assuming $h \geq k + m(k) + 3$, where $m(k)$ is the integral part of $(k-3)/2$.The instability result is obtained by showing the existence of a motion tending to $q = 0$ as $t \to +\infty$.

0. Introduction. The classical Lagrange-Dirichlet theorem states that an equilibrium configuration, say $q = 0$, of a natural Lagrangian system, with a finite number N of degrees of freedom, is stable when the potential $U(q)$ has at $q = 0$ a strict local maximum. As is well known, the converse of this criterion does not hold in general. The problem of finding additional conditions, such that the lack of a strict local maximum of the potential at an equilibrium position implies its instability, received a large amount of attention. We refer to [1, 2, 3] for surveys of results concerning this problem. Here we just recall the result by V.V. Kozlov in [1], where the instability is proved by assuming that the Lagrangian function $L = T + U \in \mathcal{C}^\infty$, and, for an integer $k \geq 3$, the k-jet of U is given by $U_{[2]} + U_{[k]}$, with $U_{[2]}$ a quadratic negative semidefinite form and $U_{[k]}$ a form of degree k which does not have a maximum at $q = 0$ in the set where $U_{[2]} = 0$. The instability is a consequence of the existence of an asymptotic motion $q(t)$ to the equilibrium position; i.e., $q(t) \to 0$ as $t \to +\infty$. Indeed, because of the energy integral, also $\dot{q}(t) \to 0$ as $t \to +\infty$, and the reversibility of the solutions of the Euler-Lagrange equations yields the instability. The asymptotic motion is obtained by Kozlov by constructing a formal series solution of the equations of motion. Then a result by Kuznetsov [4] on $\mathcal{C}^\infty$ differential systems is used, in order to prove the existence of an actual solution having the previous formal series as asymptotic expansion.

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