Differential and Integral Equations, Volume 5, Number 1, January 1992, pp. 149–164.

QUALITATIVE BEHAVIOR OF THE SOLUTIONS OF PERIODIC FIRST ORDER SCALAR DIFFERENTIAL EQUATIONS WITH WEAKLY CONCAVE NONLINEARITY

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(Submitted by Jean Mawhin)

1. Introduction. The problem to determine the number of periodic solutions to a periodic first order nonlinear differential equation

$$\dot{x} = f(t,x), \quad (t,x) \in \mathbb{R}^2, \tag{1}$$

where

$$f(t+T,x) = f(t,x), \quad (t,x) \in \mathbb{R}^2,$$

has been investigated by several authors ([1]–[6]). The related problem concerning the behavior of the non-periodic solutions has, however, only been investigated in a systematic way by Mawhin ([2]) and only for a restricted class of equations; namely, equations for which $f(t,x)$ is strictly concave in x and with $f(t,x) \to -\infty$ as $|x| \to +\infty$ for t fixed. Under these assumptions it is shown that (1) may have two, one or no periodic solutions and the behavior of the non-periodic solutions are characterized in each case. In Section 2 of the present paper, we consider the general periodic differential equation (1) and introduce the basic concepts of T-monotonic solutions and solutions of type 3. The behavior of T-monotonic solutions with unbounded intervals of definition is studied and the results are summarized in Theorem 4. In Sections 3 and 4 we generalize the results of Mawhin, assuming that $f(t,x)$ is only *weakly* concave in x excluding, however, in Section 4 the trivial case in which $f(t,x)$ is linear in x in the entire (t,x)-plane. We show that there are precisely eight different situations, illustrated in Figures 1-8. Here μ denotes the characteristic exponent for a periodic solution, and we emphasize in particular the possibility of a *band* of periodic solutions. It should also be pointed out that the vertical line segments indicate that the corresponding solution curves *either*

Received for publication September 1990.
†This author was supported by the Danish Natural Science Research Council during his visit at the Technical University of Denmark. He wishes to express his gratitude to the chairman, Professor Martin P. Bendsoe and to Professor Allan Sandqvist for this invitation and to the Mathematical Institute for its hospitality.
AMS Subject Classifications: 34C.