Differential and Integral Equations, Volume 5, Number 3, May 1992, pp. 683-691.

DEGENERATE SEMILINEAR PARABOLIC EQUATIONS

ANDREAS STAHEL
Mathematics Department, Brigham Young University, Provo, UT 84602

(Submitted by: Klaus Schmitt)

Abstract. A proof for global existence and uniqueness of solutions to degenerate parabolic equations is presented. The proof is based on an approximation by uniformly parabolic equations and an Ascoli-Arzela lemma for weak convergence. We use semigroup methods and weighted Sobolev spaces.

1. Introduction. Using elementary tools, we will prove existence and uniqueness of a global solution to the problem

$$\begin{aligned} \dot{u}(x,t) - \nabla \cdot (a(x)\nabla u(x,t)) - b(x) \cdot \nabla u(x,t) &= f(u(x,t)) && \text{in } \Omega \times I \\ u(x,t) &= 0 && \text{on } \Gamma_0 \times I \\ \frac{du(x,t)}{d\nu_a} + \beta(x)u(x,t) &= 0 && \text{on } \Gamma_1 \times I \\ u(x,0) &= u_0(x) && \text{in } \Omega \end{aligned} \tag{1}$$

on a smooth bounded subset Ω of $\mathbb{R}^N$ and $I = [0, T^+]$ where T^+ is an arbitrary positive number. The coefficient matrix $a(x)$ is assumed to be a positive definite symmetric $N \times N$ matrix, but its smallest eigenvalue might converge to zero as x approaches a singular set S in $\overline{\Omega}$. This makes the above a degenerate parabolic problem which can not be solved by the standard methods.

Linear degenerate problems of parabolic and elliptic type were considered in [3], [11], [10], [12] and [14]. Some nonlinear evolution equations are solved in [13]. In a recent series of papers, Goldstein and Lin [4], [5], [6], [7] solved similar but quasilinear problems. Their approach is based on the Crandall-Liggett theorem. A very general result for the one dimensional quasilinear case is presented in [2]. The method of weighted Sobolev spaces was applied to degenerate elliptic problems in [8], [9] and [11].

The method used in this paper is based on an approximation by uniformly parabolic problems (see [15]). We seek solutions in the natural energy space which turns out to be a weighted Sobolev space $H^1_a(\Omega)$. In the present paper, we consider a general class of degeneracies. The coefficient matrix a does not have to decay like a power of the distance to the boundary, instead we have the assumption (3). This allows also for degeneracies in the interior of the domain Ω. The papers [4], [5],

Received February 1991, in revised form June 1991.
AMS Subject Classification: 35K65.