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EXISTENCE AND NON-EXISTENCE OF POSITIVE SOLUTIONS OF NON-LINEAR ELLIPTIC SYSTEMS AND THE BIHARMONIC EQUATION

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1. Introduction. Our main objective in this paper is to give some a priori estimates and to prove existence and non-existence theorems for particular elliptic systems, which were already briefly studied in [14], and the biharmonic equation. We shall restrict ourselves here to radially symmetric solutions so that we can apply ODE-methods. It should be noted that, by a result of W.C. Troy [13], solutions of Problem (I) are automatically radially symmetric if f and g are non-decreasing functions of the dependent variables u and v. However, our results will not be restricted to such functions f and g. We consider the problem

$$\text{(I)}\qquad \begin{cases} -\Delta u = g(v), \quad v > 0 \quad \text{in } B_R, & (1.1)\\ -\Delta v = f(u), \quad u > 0 \quad \text{in } B_R, & (1.2)\\ \quad u = 0, \qquad v = 0 \quad \text{on } \partial B_R, & (1.3)\end{cases}$$

where $(u,v) \in C^2(\overline{B}_R) \times C^2(\overline{B}_R)$, with B_R a ball in $\mathbb{R}^N$ $(N \geq 4)$ of radius R. For functions f and g, the following properties are assumed to hold:

$$\text{(H1)}\qquad \begin{cases} f,\ g \in C(\mathbb{R}),\\ f(0) = 0, \quad g(0) = 0.\end{cases}$$

The system described by (I) appears in all kinds of problems in physics and chemistry because in many systems of reaction-diffusion equations, the steady-states are solutions of Problem (I). Actually, we can generalize Problem (I) to more general functions f and g; for instance, if $f(u,v)$ and $g(u,v)$ are both positive functions in u and v with certain growth-conditions. In this paper, we shall concentrate on Problem (I) in order to develop a method for proving the existence of solutions.

The idea of studying Problem (I) came from the study of the biharmonic equation with the boundary conditions $u = 0$ and $\Delta u = 0$. This problem is easily put in the form of Problem (I) when we set $g(v) = v$. This initial study brought up the idea of families of critical exponents of Problem (I), which is described in [14] in great detail. We also refer to [15] for related results on biharmonic problems. Thus in this paper we shall introduce the *critical exponents* p^* and q^* defined by

$$p^* = \frac{N-\xi}{\xi}, \quad q^* = \frac{2+\xi}{N-2-\xi},$$

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