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SOME REMARKS ON STEIN'S L log L RESULT*

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Abstract. Stein has shown that $Mf \in L \iff f \in L\log L$ where Mf is the Hardy-Littlewood maximal function. In this paper we are giving a "dual" result of the type

$$Mf \in \frac{L}{\log L} \iff f \in L,$$

and some generalizations of the related inequalities.

1. Introduction. Let Q_0 be a fixed cube in $\mathbb{R}^N$ with sides parallel to the axes and for $f \in L^1(Q_0)$ let us define the local maximal function

$$Mf(x) = \sup_{\substack{x\in Q\\ Q\subset Q_0}} \frac{1}{|Q|}\int_Q |f(y)|\,dy, \quad \forall x \in Q_0,$$

where the supremum extends over all cubes $Q \subset Q_0$ containing x with sides parallel to the coordinate axes. A well known theorem of Hardy-Littlewood asserts that if $f \in L\log L(Q_0)$, then $Mf \in L^1(Q_0)$. In [14] Stein shows that the converse of this theorem is also true; namely, it can be proved that (see, e.g., [8])

$$\int_{Q_0} |f(x)| \log\left(e + \frac{|f(x)|}{\frac{1}{|Q_0|}\int_{Q_0}|f|}\right) dx \le 2^N \int_{Q_0} Mf(x)\,dx. \tag{1}$$

In this paper (see section 3) we give the following "dual" inequality:

$$\int_{Q_0} |f(x)|\,dx \le 2^{N+1} \int_{Q_0} \frac{Mf(x)}{\log\left(e + \frac{Mf(x)}{\frac{1}{|Q_0|}\int_{Q_0} Mf}\right)}\,dx, \quad \forall f \in L^1(Q_0),\ f \not\equiv 0. \tag{2}$$

Moreover, we give a generalization of the Hardy-Littlewood theorem (see §4) which covers many well-known classical results (e.g., $\|Mf\|_{L(\log L)^\alpha} \le c\|f\|_{L(\log L)^{\alpha+1}}$ with $\alpha > 0$, $\|Mf\|_{L_p} \le c\|f\|_{L_p}$ with $p > 1$, $\|Mf\|_{L^p(\log L)^\alpha} \le c\|f\|_{L^p(\log L)^\alpha}$ with $p > 1$

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