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PERIODIC SOLUTIONS OF SOME PLANAR NON-AUTONOMOUS POLYNOMIAL DIFFERENTIAL

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To the memory of Peter Hess, who left this world too early

Abstract. This paper provides a new simple proof and some consequences of a recent existence theorem of Srzednicki for periodic solutions of some planar non-autonomous polynomial differential equations. The special case of a forced equation with a complex polynomial nonlinearity and its connection with the fundamental theorem of algebra is discussed.

1. Introduction. If $p : \mathbb{R} \to \mathbb{R}$ is a polynomial with real coefficients and odd degree, it is well known that, for every continuous function $h : [0, T] \to \mathbb{R}$, the problem

$$x' = p(x) + h(t), \quad x(0) = x(T) \tag{1}$$

has at least one solution. Observe for example that if a is the coefficient of the term of highest degree of p, then $V(x) = ax^2$ is a guiding function for (1) which satisfies all the conditions of Proposition VI.6 in [2]. When p is non-constant and of even degree, if we write

$$h(t) = \overline{h} + \tilde{h}(t),$$

with $\overline{h}$ the mean value of h over $[0, T]$, then it follows from Theorem 1 in [3] that there exists $h_0 \in \mathbb{R}$ such that (1) has at least one solution for $\overline{h} = h_0$, no solution for $\overline{h}$ in one of the open half-lines with extremity h_0 and at least two solutions for $\overline{h}$ in the other open half-line.

In the special case where $h = 0$, every possible solution x of (1) is such that

$$\int_0^T (x'(t))^2 \, dt = \int_0^T p[x(t)]x'(t) \, dt = 0,$$

and therefore is a constant function, whose value x_0 is the solution of the algebraic equation $p(x) = 0$. Another way to reach the same conclusion is to notice that (1) with $h = 0$ is a gradient system. The first existence result for periodic solutions may be viewed therefore as an extension of the elementary theorem in algebra which insures

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