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DISTRIBUTIONAL AND SMALL SOLUTIONS FOR LINEAR TIME-DEPENDENT DELAY EQUATIONS

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Abstract. In this paper, the relation between distributional and small solutions of linear time-dependent delay equations is studied. A class of infinite order distributional solutions is defined such that the existence of a small solution implies the existence of a distributional solution in this class. A sufficient condition for the nonexistence of small solutions is proved.

Introduction. Consider a linear time-dependent delay equation

$$\dot{x}(t) = A(t)x(t) + B(t)x(t-1), \qquad t \geq s, \tag{0.1}$$

where $A(\cdot)$ and $B(\cdot)$ are real continuous $n \times n$ matrix functions. To define a solution of (0.1) one has to specify an initial condition on the interval $[s-1, s]$. Let $x_t(\theta) = x(t+\theta)$ for $-1 \leq \theta \leq 0$ and $\mathcal{C} = C[-1,0]$ be the Banach space of continuous functions on $[-1,0]$ provided with the supremum norm.

The initial value problem (0.1) with $x_s = \varphi$ and $\varphi \in \mathcal{C}$ is well-posed, that is, for any given $\varphi \in \mathcal{C}$ a solution $x = x(\cdot;\varphi)$ exists and is unique. Furthermore, the solution can be obtained using the method of steps: On the interval $[s, s+1]$ equation (0.1) reduces to an inhomogeneous ODE and can be solved, then using the solution on $[s, s+1]$ equation (0.1) reduces to an inhomogeneous ODE on the interval $[s+1, s+2]$ etc. In particular, if $B(\cdot)$ is smooth, the solution becomes smoother with increasing time.

A solution x of (0.1) is called *small* if

$$\lim_{t\to\infty} x(t)e^{kt} = 0 \quad \text{for every} \quad k \in \mathbb{R}. \tag{0.2}$$

The zero solution is a trivial small solution. Small solutions that are not identically zero are called nontrivial. In an earlier paper [11] we have studied the existence

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