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A REMARK ON THE EQUALITY $\det Df = \operatorname{Det} Df$*

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Abstract. We prove equality between the Jacobian and the so-called *distributional* or *weak determinant* introduced by J. Ball [1]. This result is slightly more general than the one recently obtained in [10].

1. Introduction. For Ω a domain of $\mathbb{R}^N$ and $f\colon \Omega \to \mathbb{R}^N$, $f = (f^1, \cdots, f^N)$, a mapping of Sobolev class $W^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$, we denote by $Df(x)\colon \mathbb{R}^N \to \mathbb{R}^N$ $(x \in \Omega)$ the differential and by $J(x; f) = \det Df(x)$ the Jacobian. A mapping $f \in W^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ is said to be *orientation preserving* if $J(x; f) \geq 0$, for almost every $x \in \Omega$.

Let $f \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ with $p \geq N^2/(N+1)$; the weak Jacobian of f is the Schwartz distribution $J_f = \operatorname{Det} Df \in \mathcal{D}'(\Omega)$ defined by the rule

$$\langle J_f, \varphi \rangle = -\int_\Omega f^N(x)\, J(x; (f^1, \dots, f^{N-1}, \varphi))\, dx,$$

for any test function $\varphi \in C_0^\infty(\Omega)$. In [13] S. Müller proved a conjecture of J. Ball that, if $\operatorname{Det} Df \in L^1$, then $\det Df = \operatorname{Det} Df$; i.e.,

$$\int_\Omega \varphi(x)\, J(x; f)\, dx = -\int_\Omega f^N(x)\, J(x; (f^1, \dots, f^{N-1}, \varphi))\, dx, \tag{1.1}$$

for any test function $\varphi \in C_0^\infty(\Omega)$. In [14] it is proved that, if $f \in W^{1,N}(\Omega; \mathbb{R}^N)$ is an orientation preserving mapping, then the Jacobian J belongs locally to the Zygmund class $L \log L$; for another proof of this result, see [8]. On the other hand, in [10] an almost optimal condition for the integrability of the Jacobian of an orientation preserving mapping is given; in particular, it is proved that if f is an orientation preserving mapping and $|Df| \in \frac{L^N}{\log L}$ (see Section 2), then $J \in L^1_{\mathrm{loc}}$ and the following uniform bound holds:

$$⨍_B J(x; f)\, dx \leq c(N) ⨍_{3B} \frac{|Df(x)|^N}{\log\left(e + \frac{|Df(x)|}{|Df|_{3B}}\right)}, \tag{1.2}$$

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