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ENERGY AND REGULARITY INEQUALITIES FOR VOLTERRA EQUATIONS OF PARABOLIC TYPE

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Abstract. This paper presents an energy inequality and a regularity inequality for the Volterra equation of parabolic type. They correspond to the energy equality and the regularity inequality for the evolution equation of parabolic type, respectively.

0. Introduction. Let $W(dt)$ be the measure on $[0, \infty)$ so that $W([0, t])$ satisfies the integral equation

$$a\, W([0,t]) + \int_0^t k(t-s) W([0,s])\, ds = t, \quad t \geq 0 \tag{0.1}$$

for some constant $a \geq 0$, and non-negative and non-increasing function $k \in L^1_{\text{loc}}$ $([0, \infty))$ with $k(\infty) = 0$. In addition, suppose that $k(0+) = \infty$ if $a = 0$. Such a measure satisfying $W(\{t\}) = 0$ for $t \geq 0$ exists uniquely (see [10, p. 64], [9, Proposition]).

In the present paper we are concerned with the Volterra equation of parabolic type

$$u(t) = \phi - \int_{[0,t]} Lu(t-s) W(ds), \quad t \geq 0. \tag{VE}$$

Here L is a non-negative and self-adjoint operator on a Hilbert space X with the norm $\|\cdot\|$, and ϕ belongs to the domain $D(L)$ of L. The existence and the uniqueness of the solution of (VE) has been studied by many authors ([3], [4], [5], [6], [9], [10] etc.). Here a function u is called a *solution* of (VE) if (i) $u(t)$, $t \geq 0$, belongs to $D(L)$, (ii) u and Lu belong to $C([0, \infty); X)$ with $\|u(t)\| + \|Lu(t)\| \leq Ce^{\rho t}$, $t \geq 0$, for some $C > 0$, $\rho \geq 0$, and (iii) u satisfies (VE) for every $t \geq 0$. Note that when $k \equiv 0$, (VE) is the integral form of the evolution equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t) = -Lu(t)/a, & t > 0, \\ u(0) = \phi. \end{cases} \tag{EE}$$

The aim of the present paper is to derive an energy inequality and a regularity inequality for (VE), which correspond to the energy equality and the regularity inequality for (EE), respectively.

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