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EXISTENCE OF SOLUTIONS OF SEMILINEAR ELLIPTIC PROBLEMS ON UNBOUNDED DOMAINS

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Abstract. In this paper we study the existence of solutions of the semilinear elliptic problem $-\Delta u + \lambda u = |u|^{p-2}u$ on some classes of unbounded domains.

1. Introduction. Let $N \geq 2$ and $2 < p < 2^*$, where $2^* = \frac{2N}{N-2}$ for $N \geq 3$, $2^* = \infty$ for $N = 2$. Throughout this paper, we will study the existence of positive solutions of the semilinear elliptic equation

$$\begin{cases} -\Delta u + \lambda u = |u|^{p-2}u & \text{in } \Omega \\ u \in H_0^1(\Omega), \end{cases} \tag{1.1}$$

where Ω is an unbounded domain in $\mathbb{R}^N$, and $\lambda \in \mathbb{R}^+$.

Associated with equation (1.1), we consider the energy functional on $H_0^1(\Omega)$:

$$F(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + \lambda u^2) - \frac{1}{p}\int_\Omega |u|^p$$

and the constraint energy functional $f(u) = \int_\Omega (|\nabla u|^2 + \lambda u^2)$ on the manifold $\{u \in H_0^1(\Omega) \mid \int_\Omega |u|^p = 1\}$. Let

$$\alpha(\Omega) = \inf\left\{ \int_\Omega (|\nabla u|^2 + \lambda u^2) \mid u \in H_0^1(\Omega), \int_\Omega |u|^p = 1 \right\}. \tag{1.2}$$

Then we say that $\alpha(\Omega)$ or Ω admits a minimizer if there is $u \in H_0^1(\Omega)$ satisfying

$$\int_\Omega (|\nabla u|^2 + \lambda u^2) = \alpha(\Omega), \quad \text{and} \quad \int_\Omega |u|^p = 1.$$

In Section 2 we develop several tools for general domains (bounded or unbounded), which will be used later. It is known that given a minimizing sequence $\{u_n\}$ of some energy functional which is lower semicontinuous and bounded below, we may apply the Ekeland Variational Principle to get a Palais-Smale (or briefly $(\mathrm{PS})_c$) sequence $\{v_n\}$ such that $\|u_n - v_n\| < \frac{1}{n}$ for each n (see Brezis-Nirenberg ([5])). Though the energy functional F is not bounded below, however in Theorem 2.1 we prove that for

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