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BOUNDEDNESS OF TRAJECTORIES OF PARABOLIC EQUATIONS AND STATIONARY SOLUTIONS VIA DYNAMICAL METHODS

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To the memory of Peter Hess

Abstract. We consider an initial boundary value problem for the parabolic equation $u_t = \Delta u + f(x, u, \nabla u)$ in a bounded domain in $\mathbb{R}^N$. Assuming that the initial value belongs to the boundary of the domain of attraction of a stable equilibrium we derive sufficient conditions for global existence and boundedness of the solution. The main assumption is the existence of a Lyapunov functional "at infinity" satisfying a kind of the Palais-Smale condition. Applications to existence proofs of (multiple) unstable solutions of superlinear elliptic boundary value problems are given.

1. Introduction. In this paper we are concerned with dynamical systems generated by certain parabolic equations. We assume existence of a Lyapunov functional E "at infinity" and we study global existence and boundedness of trajectories starting on the boundary of the domain of attraction D_A of a given stable equilibrium u_s. This study is motivated by the fact that the boundedness of these trajectories gives us better information on the domain of attraction D_A (see [19, Theorem 5.4]) and, moreover, it can often be used to prove existence of unstable equilibria. Such existence results can sometimes be obtained also by other methods (mountain pass theorem, Ljusternik-Schnirelman theory, topological degree) however, the dynamical approach seems to yield additional information also, e.g. concerning connecting orbits.

A model problem for our study is the following equation

$$\begin{aligned} u_t =& \Delta u + \tilde{f}(x, u, \nabla u) + f(x, u), && t \in (0, +\infty),\ x \in \Omega \\ u =& 0, && t \in (0, +\infty),\ x \in \partial\Omega \\ u(0, x) =& u_o(x), && x \in \bar{\Omega} \end{aligned} \tag{P}$$

where Ω is a bounded domain in $\mathbb{R}^N$ and $\tilde{f}$ is "small" in some reasonable sense. The corresponding Lyapunov functional E for (P) is then given in the following way:

$$E(u) = \int_\Omega \Big(\frac{1}{2}|\nabla u|^2 - F(x, u)\Big)\, dx,$$

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