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REMARK ON THE PAPER BY S. DUBOIS "MILD SOLUTIONS TO THE NAVIER-STOKES EQUATIONS AND ENERGY EQUALITY"

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S. Dubois obtains an interesting theorem which states that every mild solution v to the Navier-Stokes equations in the class $C([0,T);L^3(\mathbb{R}^3))$ with the initial data $v_0 \in L^2(\mathbb{R}^3)\cap L^3(\mathbb{R}^3)$, div $v_0 = 0$ belongs, necessarily to $C([0,T);L^2(\mathbb{R}^3))$ and fulfills energy equalities

$$\|v(t)\|_{L^2}^2 + 2\int_s^t \|\nabla v(\tau)\|_{L^2}^2 d\tau = \|v(s)\|_{L^2}^2 \quad \text{for all } 0 \le s \le t < T. \tag{0.1}$$

In particular, v is in the Leray-Hopf class. She first establishes a certain bilinear estimate for the evolution $\int_0^t e^{(t-\tau)\Delta}P(v\cdot\nabla v)(\tau)d\tau$, and then investigates behavior near $t\to +0$ of $v(t)$ to apply her estimate to the proof of energy inequalities. See the main result Theorem 2.1 in her paper. The purpose of this note is to show that one can prove validity of energy equalities directly by a local existence theorem of the strong solution with the aid of uniqueness of mild solutions in the class $C([0,T);L^3(\mathbb{R}^3))$. It is essential for our argument that the local existence time of the strong solution can be taken *uniformly* for every *precompact* subset in $L^3(\mathbb{R}^3)$ of the initial data. This was implicitly pointed out by Brezis [1]. Our theorem holds also for $\mathbb{R}^n$ with $n\ge 3$. We follow the same notations and definitions of Dubois' paper except for $n \ge 3$. For simplicity, we denote by $L^n_\sigma(\mathbb{R}^n)$ the solenoidal vector fields in $L^n(\mathbb{R}^n)$.

Theorem 1. *Let $v_0 \in L^2(\mathbb{R}^n)\cap L^n_\sigma(\mathbb{R}^n)$. Suppose that v is a mild solution of (NSI) in the class $C([0,T);L^n(\mathbb{R}^n))$ arising from v_0. Then v is in $C([0,T);L^2(\mathbb{R}^n))$ and fulfills energy equalities* (0.1). *In particular, v is in the Leray-Hopf class on* $(0,T)$, *i.e.,*

$$v \in L^\infty(0,T;L^2(\mathbb{R}^n))\cap L^2(0,T;\dot{H}^1(\mathbb{R}^n)).$$

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