Differential and Integral Equations, Volume 3, Number 6, November 1990, pp. 1001–1014.

THE MAXIMUM PRINCIPLE FOR SEMICONTINUOUS FUNCTIONS

MICHAEL G. CRANDALL†
Department of Mathematics, University of California, Santa Barbara, California 93106

HITOSHI ISHII
Department of Mathematics, Chuo University, Bunkyo–ku, Tokyo 112, Japan

Abstract. The result of calculus which states that at a maximum of a twice differentiable function the gradient vanishes and the matrix of second derivatives is nonpositive plays a significant role in the theory of elliptic and parabolic differential equations of second order, where it is used to establish many results for solutions of these equations. The theory of viscosity solutions of fully nonlinear degenerate elliptic and parabolic equations, which admits nondifferentiable functions as solutions of these equations, is now recognized to depend on a "maximum principle" for semicontinuous functions, which replaces the calculus result mentioned above. This work contains a more general statement of this form together with a simpler proof than were available heretofore.

0. Introduction. The term "the maximum principle" is a rather vague expression which stands for a remarkably rich collection of results and methods which permeate the theory of second order elliptic and parabolic equations. Classical implementations of the maximum principle rely on the result of elementary calculus which states that if z is a local maximum (respectively, minimum) point of a twice differentiable function u, then the gradient of u at z, $Du(z)$, vanishes while the matrix of second derivatives $D^2u(z) = \big(u_{x_ix_j}(z)\big)$ is nonpositive (nonnegative) in the usual ordering of symmetric matrices. It has gradually become clear that analogues of this result which remain valid for functions which are merely semicontinuous play the same role in the theory of viscosity solutions of fully nonlinear second order elliptic and parabolic equations as the calculus result plays in more classical situations. The current paper provides fundamental information concerning the maximum principle for semicontinuous functions.

Indeed, our goal is to extend the result of M. Crandall [3] concerning certain "second derivatives" of semicontinuous functions (a sharpening of results born in H. Ishii [6] and P.L. Lions and H. Ishii [7]) in such a way as to allow one to simplify all of the theory of viscosity solutions of second order equations outlined in [7] as the

---

Received June 20, 1990.
†Supported in part by the following grants and contracts: Army Research Office DAAL03-87-K-0043, National Science Foundation DMS-8505531, Office of Naval Research N00014-88-K-0134.
AMS Subject Classifications: 35B50, 26B05, 35D99.