Differential and Integral Equations, Volume 3, Number 5, September 1990, pp. 973–978.

ON A QUESTION OF BLOW-UP FOR SEMILINEAR PARABOLIC SYSTEMS

JEFF MORGAN†
Department of Mathematics, Texas A&M University, College Station, TX 77843, USA

(Submitted by: Glenn Webb)

Abstract. We discuss the question of whether the addition of diffusion and no-flux boundary conditions to a reaction mechanism can cause finite time blow-up. We give an example with memory for which this is the case, and give an example without memory which might stimulate further work in this area.

1. Introduction. A long-standing question in the theory of differential equations asks whether the addition of diffusion in a reaction mechanism can cause blow-up in finite time. More specifically, does there exist a locally Lipschitz vector field $f: \mathbb{R}^m \to \mathbb{R}^m$, a positive definite diagonal $m \times m$ matrix D and a smooth function $u_0: [0,1] \to \mathbb{R}^m$, such that for all $y_0 \in \mathbb{R}^m$, solutions of

$$y'(t) = f(y(t)), \quad y(0) = y_0, \quad t \geq 0 \tag{1.1}$$

exist for all time, but the solution of

$$\begin{aligned} u_t &= Du_{xx} + f(u), && t > 0, \ 0 < x < 1 \\ u_x &= 0, && t > 0, \ x = 0, 1 \\ u &= u_0, && t = 0, \ 0 < x < 1 \end{aligned} \tag{1.2}$$

blows-up in finite time? When certain invariant region or Lyapunov-type assumptions are made, then the question has been resolved to the negative (cf. Bates [1], Hollis, Martin, Pierre [4], Morgan [5]). Also, when $m = 1$ or monotonicity assumptions are imposed on f, then the theory of super or sub solutions can be used to resolve this question to the negative (cf. Czischke [2], Pao [6], Hollis [3]). The only positive results to date are those for which boundary feed terms (via nonhomogeneous Dirichlet boundary conditions) replace the no-flux boundary conditions for select components of (1.2) (cf. Hollis [3]). Of course, in this case, solutions of (1.1) are no longer solutions of (1.2), and the boundary feed mechanism aids the finite time blow-up.

Perhaps one reason why this question remains unresolved is a lack of understanding of the fragile nature of global existence. In a recent conversation with R.H.

Received March 13, 1989, in revised May 26, 1989.
†Supported in part by NSF grant #DMS 8813071.
AMS Subject Classifications: 34A15, 34C99, 35B99, 35K50.