Differential and Integral Equations, Volume 3, Number 5, September 1990, pp. 933–943.

THE LAST FREE COEXISTENCE-LIKE PROBLEM

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(Submitted by: Luigi Salvadori)

Abstract. This paper deals with the Liapunov stability of the origin for the system

$$\ddot{x} + xf(x) = 0, \quad \ddot{y} + yw(x) = 0, \quad x, y \in \mathbb{R}, \quad f(0) > 0. \tag{*}$$

If there exists $s(x, \dot{x})$ such that $\dot{y}s - y\dot{s}$ is a first integral, and some smoothness and nondegeneracy conditions hold, then the stability is equivalent to "coexistence" of periodic solutions of every Hill's equation in a certain family.

Given the functions s and f, there exists at most one function w such that the system $(*)$ admits $\dot{y}s - y\dot{s}$ as first integral, but generally no such w exists. Certain special functions s have the property that w can be found in connection with each f so that $(*)$ has the first integral $\dot{y}s - y\dot{s}$ (an example is $s(x, \dot{x}) = x$ where we can choose $w = f$). Each of these special functions s generates the following problem: determine all the functions f such that the origin is a stable equilibrium for $(*)$ with w defined by s and f. We call such problems *free coexistence-like.*

Some previous papers solved all the free coexistence-like problems except the one generated by $s(x, \dot{x}) = x\dot{x}$ which is solved in this paper.

1. Introduction. This paper deals with a problem in Stability of the Equilibrium for a point in the plane driven by a certain kind of purely positional force. Other general features of the system are that there is neither conservation nor dissipation of energy (the force is not a gradient), and one of the two degrees of freedom is separate from the other, that is, one of the two equations is independent of the other.

The exact statement of the problem may look a little odd without proper motivation. So we start with some background (see [1] and [2] for more details).

Let f, w be real functions, defined in some open interval of $\mathbb{R}$ containing 0, and consider the system of two ordinary differential equations

$$\ddot{x} + xf(x) = 0, \quad \ddot{y} + yw(x) = 0, \quad x, y \in \mathbb{R}, \quad f(0) > 0, \quad f \in C^1, \quad w \in C^0. \tag{1.1}$$

Received April 4, 1989.

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A Paola Calderoni della cui intelligenza matematica e della cui simpatia sentiamo la mancanza.

AMS Subject Classifications: 34D20.