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VARIATIONAL METHODS FOR NONLINEAR EIGENVALUE INEQUALITIES

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Abstract. This paper deals with the existence of solutions for nonlinear eigenvalue problems associated with multi-valued operators. The main result extends the Lagrange multipliers theory developed by F. Browder and its proof is based on direct variational methods.

1. Introduction.

1.1. The Problem. Several physical problems possessing multiple solutions (for example, the buckling and vibration of structures, see [1, 2]) have been modeled by eigenvalue inequalities of the type:

$$\langle A(u) - \lambda B(u), v - u\rangle \geq 0, \quad \forall v \in K, \tag{1}$$

where A and B are two possibly nonlinear operators in some function space $\mathcal{U}$. The set $K \subset \mathcal{U}$ is closed and convex in $\mathcal{U}$ and $\langle\cdot,\cdot\rangle$ denotes a convenient scalar product.

Whenever K is the whole space $\mathcal{U}$, the inequality (1) reduces to the following equation:

$$A(u) - \lambda B(u) = 0. \tag{2}$$

Variational methods can be employed to study (1) or (2) whenever A and B are potential operators; that is to say, if there exist two real functions F and G whose gradients are A and B respectively.

The most important result concerning the first eigensolution of the equation (2) can be found in Browder's paper [3] where the following result is proved:

"Let $\mathcal{U}$ be a linear normed space and $F, G, : \mathcal{U} \to \mathbb{R}$ be two Fréchet differentiable real functionals. If F assumes its minimum restricted to the set $\{v \in \mathcal{U} : G(v) = G(u)\}$ at the point u and if $G'(u) \neq 0$, then there exists $\lambda \in \mathbb{R}$ such that $F'(u) = \lambda G'(u)$."

The idea of Browder's proof is to consider all possible trajectories $f(u,t) = u + tv_0 + pv_1$ departing from the solution u of the minimization problem, where v_0 is any element orthogonal to $G'(u)$ and v_1 is such that $\langle G'(u), v_1\rangle = 1$. Choosing the small real parameter p in such a way that $G(f) = G(u)$, it becomes necessary that

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