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GENERALIZED WEIGHTS FOR ORTHOGONAL POLYNOMIALS

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Dedicated to Professor Hokee Minn on the occasion of his retirement

Abstract. In this paper we obtain the real weights with respect to which the polynomial solutions of second-order differential equations are orthogonal. Depending on the nature of the supports of these weights, there are four distinct types; Jacobi, Bessel, Laguerre, and Hermite, which have their supports on a bounded interval, a singleton, a half line and the whole line respectively. Weights are initially sought in the space of hyperfunctions but it turns out that all are distributions except the weight for the Bessel type, which is a hyperfunction (in fact, an ultradistribution of Gevrey class) with a point support. This latter result resolves the problem (raised by R.D. Morton and A.M. Krall) of finding a real weight and interval for which Bessel polynomials are orthogonal.

1. Introduction. In [8], R.D. Morton and A.M. Krall introduced distributional weights for orthogonal monic Chebychev polynomials $p_n(x)$ (for definition, see [6, 8]) associated with a given sequence of real numbers $\{\mu_n\}_0^\infty$, called moments, satisfying

$$\Delta_n = \det[\mu_{i+j}]_{i,j=0}^n \neq 0, \quad n = 0, 1, 2, \dots. \tag{1.1}$$

Their method is to expand the weight $w(x)$ into a formal functional Taylor series

$$w(x) = \sum_{n=0}^{\infty} (-1)^n \mu_n \delta^{(n)}(x)/n! \tag{1.2}$$

and show that it can be extended, through Fourier transformation, to a continuous linear functional on the space of slowly increasing test functions assuming suitable estimates on the moments and on the Fourier transform of $w(x)$. The method recovers and extends the classical results for Jacobi, Laguerre and Hermite polynomials but fails to work in the case of Bessel polynomials (cf. [7]) which is an example motivating the work [8]. Since the δ-expansion (1.2) of $w(x)$ is only formal in the distribution sense (unless the series terminates, which is not acceptable in the framework of this paper) and a real weight for the Bessel polynomials must have support at $\{0\}$, it is impossible to find such a weight in the space of distributions.

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