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ON THE GLOBAL WELL-POSEDNESS OF THE BENJAMIN-ONO EQUATION

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Abstract. The Cauchy problem for the Benjamin-Ono equation

$$\partial_t u + u\partial_x u + H\partial_x^2 u = 0$$

is considered. It is shown that this problem is globally well-posed in $H^s(\mathbb{R})$ for any $s \geq 3/2$. It is also established that for such values of s, local and global smoothing effects are present in the solution. These smoothing effects which are the main tools in the proof of the extremal case ($s = 3/2$) are reminiscent of the dispersive character of the associated linear equation.

1. Introduction. In this paper we consider the Cauchy problem for the Benjamin-Ono (BO) equation

$$\begin{cases} \partial_t u + u\partial_x u + H\partial_x^2 u = 0 & x, t \in \mathbb{R} \\ u(x,0) = u_0(x) \end{cases} \tag{1.1}$$

where $u = u(x,t)$, $\partial_t = \frac{\partial}{\partial t}$, $\partial_x = \frac{\partial}{\partial x}$, and H is the Hilbert transform; i.e.,

$$Hf(x) = pv\frac{1}{\pi}\int \frac{f(y)}{x-y}\,dy = F^{-1}(i \cdot sgn(\xi)\hat{f}(\xi))$$

with $\hat{}$ and F^{-1} denoting the Fourier transform and its inverse respectively.

The BO equation (T.B. Benjamin [2], and H. Ono [24]) arises in the study of uni-directional propagation of nonlinear dispersive waves, and presents the interesting fact that the operator modelling the dispersive effect is not local.

Our aim is to investigate the global well-posedness of the problem (1.1) in the classical Sobolev spaces $H^s(\mathbb{R})$, and the regularity of their solutions in other function spaces. Our notion of well-posedness contains: existence, uniqueness, persistence property (i.e., the solution $u(t)$ at any time $t \in [-T,T]$ belongs to the same space X as does the initial data u_0, and describes a continuous curve in X), and continuity of the solution as a function of the initial data (i.e., continuity of the map $u_0 \to u(t)$ from X to $C([-T,T] : X)$). When $T = T(\| u_0 \|_X) < \infty$ it is said that the problem

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