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SELECTIONS OF LIPSCHITZ MULTIFUNCTIONS GENERATING A CONTINUOUS FLOW

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1. Introduction. Given a Hausdorff continuous multifunction F with compact values, solutions of the Cauchy problem

$$\dot{x}(t) \in F(t, x(t)), \qquad x(t_0) = x_0, \tag{1.1}$$

can be obtained [3] by constructing a directionally continuous selection f and solving

$$\dot{x}(t) = f(t, x(t)), \qquad x(t_0) = x_0. \tag{1.2}$$

In the case where F is Lipschitz continuous with respect to x, we prove in this paper that a selection f can be found with the additional property that, for any t_0, x_0, the problem (1.2) has a unique solution, depending continuously on the initial data. The discontinuous (time dependent) vector field f thus generates a continuous flow, whose integral curves are also trajectories of (1.1). This provides an alternative proof to a theorem of Cellina [8], asserting the existence of a family of solutions of (1.1) which vary continuously with the initial point x_0. Of course, when all images $F(t, x)$ are convex our result is well known, because of the existence of a Lipschitz continuous selection.

We remark that, if one could construct a selection whose directional variation is locally bounded, then the uniqueness and Lipschitz continuous dependence of the solution of (1.2) would follow from [4, 6]. Unfortunately, it is not clear whether this approach can be implemented: here the selection f is constructed as the uniform limit of a sequence of piecewise Lipschitz continuous approximate selections f_ν, whose directional variation may well approach infinity as $\nu \to \infty$. The uniqueness property is proved by a somewhat weaker argument, which does not yield a Lipschitz or Hölder modulus of continuity.

2. The main result. For the basic definitions and properties of multifunctions we refer to [1]. We say that a map $f : \mathbb{R}^{n+1} \mapsto \mathbb{R}^n$ is *directionally continuous* if, for a given cone $\Gamma \subset \mathbb{R}^{n+1}$, one has

$$f(t, x) = \lim_{\nu \to \infty} f(t_\nu, x_\nu)$$

for every t, x and every sequence $(t_\nu, x_\nu) \to (t, x)$ with $(t_\nu - t,\ x_\nu - x) \in \Gamma$ for each ν. Solutions of differential equations with directionally continuous right hand side were studied in [3–7, 9]. Aim of this paper is to prove the following:

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