Differential and Integral Equations, Volume 3, Number 4, July 1990, pp. 799–800.

A NOTE ON PALAIS–SMALE CONDITION AND COERCIVITY

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Introduction. It has been observed ([2], [3]) that, for a C^1 function bounded from below on a Banach space, the Palais-Smale condition implies coercivity. Our aim in this note is to generalize this result under weaker regularity assumptions and to consider the coercivity of $|\phi|$ when ϕ is not bounded from below.

Our main tool is Ekeland's variational principle which we recall now:

Theorem 1. [1]. *Let X be a complete metric space and let $\phi : X \to (-\infty, \infty]$ be a lower semi-continuous function such that $\inf_X \phi \in \mathbb{R}$. Let $\epsilon > 0$ and $u \in X$ be given such that $\phi(u) \leq \inf_X \phi + \epsilon$. Then, for every $\lambda > 0$, there exists $v \in X$ such that*

i) $\phi(v) \leq \phi(u)$

ii) $d(u,v) \leq 1/\lambda$

iii) $\phi(w) > \phi(v) - \lambda\epsilon d(w,v)$, $\forall\, w \neq v$.

1. Coercivity of ϕ. Let X be a Banach space. A Gateaux differentiable function $\phi : X \to \mathbb{R}$ satisfies the Palais-Smale condition if every sequence (u_n) such that $(\phi(u_n))$ is bounded and $\phi'(u_n) \to 0$ contains a convergent subsequence. A function $\phi : X \to \mathbb{R}$ is coercive if $\phi(u) \to +\infty$ as $|u| \to \infty$.

Theorem 2. *Let X be a Banach space and let $\phi : X \to \mathbb{R}$ be a Gateaux differentiable lower semi-continuous function satisfying the Palais-Smale condition. If ϕ is bounded from below, then ϕ is coercive.*

Proof: Assume on the contrary that

$$c = \varliminf_{|u| \to \infty} \phi(u) \in \mathbb{R}.$$

Received January 13, 1989
AMS Subject Classifications: 58F05, 49A27.