Differential and Integral Equations, Volume 3, Number 4, July 1990, pp. 601–616.

QUADRATIC GROWTH OF SOLUTIONS OF FULLY NONLINEAR SECOND ORDER EQUATIONS IN $\mathbb{R}^n$

MICHAEL G. CRANDALL

Department of Mathematics, University of California, Santa Barbara, California 93106 USA

P.L. Lions

Ceremade, Université Paris-Dauphine, Place de Lattre de Tassigny, 75775 Paris Cedex 16, France

Abstract. Fully nonlinear (degenerate) elliptic and parabolic equations of second order in $\mathbb{R}^N$ of the forms $u + F(D^2u) = f(x)$ and $u_t + F(D^2u) = 0$, where F is a nonincreasing function from the symmetric $N \times N$ matrices equipped with their usual ordering to $\mathbb{R}$, are considered. Existence and uniqueness theorems are proved in the class of solutions of quadratic growth when the data (f in the elliptic case and the initial data in the parabolic case) have suitable properties. In the parabolic case, a semiflow is obtained and the "inverse problem" of determining properties of similar flows necessary and sufficient to guarantee that they are the time t maps for such an equation is solved.

Introduction. The simplest form the Cauchy problem for a fully nonlinear (possibly degenerate) parabolic equation may have is

$$u_t + F(D^2u) = 0 \quad \text{on} \quad (0,\infty) \times \mathbb{R}^n, \quad u(0,x) = \psi(x) \quad \text{for } x \in \mathbb{R}^n, \tag{CP}$$

in which $\mathcal{S}^n$ is the set of real symmetric $n \times n$ matrices equipped with its usual order and $F : \mathcal{S}^n \to \mathbb{R}$ is continuous and satisfies the "ellipticity" condition

$$F(B) \leq F(A) \quad \text{when } A, B \in \mathcal{S}^n \quad \text{and} \quad A \leq B. \tag{0.1}$$

We are interested in results concerning (CP) which place no further restrictions on F beyond continuity and (0.1); that is, we seek to determine a natural large space of functions on $\mathbb{R}^n$ such that for each ψ in this space (CP) has a solution u on $[0,\infty) \times \mathbb{R}^n$ satisfying conditions which guarantee its uniqueness. Our resolution of this problem involves the space $QUC(\mathbb{R}^n) = Q(\mathbb{R}^n) + UC(\mathbb{R}^n)$ of sums of pure quadratic functions on $\mathbb{R}^n$ (the elements of $Q(\mathbb{R}^n)$) and uniformly continuous functions on $\mathbb{R}^n$ (the elements of $UC(\mathbb{R}^n)$). Theorem 2.4 below asserts that for each $\psi \in QUC(\mathbb{R}^n)$, (CP) has a unique solution u which grows at most quadratically in x and that

Received December 31, 1989.
AMS Subject Classifications: 35B05, 35B50, 35J60, 35K55.
Supported in part by the following grants and contracts: Army Research Office DAAL03-87-K-0043 and DASA 45-88-C009, National Science Foundation DMS-8505531, Office of Naval Research N00014-88-K-0134.