Differential and Integral Equations, Volume 2, Number 3, July 1989, pp. 363-377.

A DIFFERENTIABILITY RESULT FOR THE RELATIVE REARRANGEMENT

J.M. RAKOTOSON

Laboratoire d'Analyse Numérique, Université Lyon 1
Bât 101, 43, bd du 11 Novembre 1918, 69622 Villeurbanne Cedex, France

(Submitted by: Roger Temam)

Motivation – Introduction. For solving the following plasmas physics model:

$$(P_1) \quad \begin{cases} -\Delta u + \lambda \dfrac{d^2 u_*}{ds^2}(\beta(u)(x)) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

R. Temam [10] proposes to formulate the problem (P_1) as the Euler equations of the following variational problem

$$(P_2) \text{ Minimize} \qquad J(v) = \int_\Omega |\nabla v|^2 \, dx + \lambda \int_{\Omega^*} |\frac{dv_*}{ds}|^2 \, ds - 2 \int_\Omega f v \, dx$$

over the set $K = \{v \in H_0^1(\Omega),\ v_* \in H^1(\Omega^*)\}$. Here, $\Omega^* = (0, \operatorname{meas} \Omega)$, v_* is the decreasing rearrangement of v.

But, from the results of Sperner [9], R. Temam and the author [8], it is known that if $u \in W^{1,p}(\Omega)$ then u_* is only in $W^{1,p}_{\text{loc}}(\Omega^*)$. More precisely, it is proved in [8] (see also [7]) that if Ω belongs to a class of "smooth" sets Σ_i, then

$$|k(\cdot)\frac{du_*}{ds}|_{L^p(\Omega^*)} \leq Q(\Omega) \cdot |\nabla u|_{L^p(\Omega)}. \tag{1}$$

So it is natural to ask the following questions:

(Q_1) Does the inequality (1) hold for a class of sets other than Σ_i?

(Q_2) Does the smoothness of the domain really interfere in the regularity of u_*?

(Q_3) In view of using the set K, can we remove the singularity at $s = 0$ or $s = |\Omega|$ for u_*?

In the first section, we will answer the question (Q_1) by proving that if inequality (1) is valid for all u in $W^{1,1}(\Omega)$, then Ω belongs to Σ_i.

In the second section, we will exhibit some counterexamples showing the necessity of the smoothness of the domain. We begin the last section by showing that if Ω is in a class of sets Σ_i, u in $W^{1,p}(\Omega)$, $p > N$ then u_* is in $W^{1,q}(\Omega^*)$ for $1 \leq q < 1/(1 + \frac{1}{p} - \frac{1}{N})$. We show by counterexample that result is sharp for the class of sets Σ_i.

Received September 13, 1988.
AMS Subject Classifications: 35J60, 26A24, 26A99, 46E30, 28A99.