

SMALLEST LYAPUNOV FUNCTIONS OF DIFFERENTIAL INCLUSIONS

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(Submitted by: Klaus Deimling)

Abstract. We provide a first answer to the question: given a differential inclusion, does there exist a *smallest* nonnegative extended lower semicontinuous (i.e., take their values in $\mathbb{R}_+ \cup \{+\infty\}$) Lyapunov function *larger than a given* lower semicontinuous function? Since the lower semicontinuous functions involved in the statement of this problem are not necessarily differentiable, we have to weaken the usual definition of a derivative and replace it by the one of epicontingent derivative. This allows us to characterize lower semicontinuous Lyapunov functions of a differential inclusion. With this definition at hand, we shall answer this question. These results find natural applications in differential games.

The tool for achieving this objective is the existence of largest closed viability (and/or invariance) domains of a differential inclusion contained in a given closed subset. Hence, we shall provide in the appendix the proof of their existence as well as the division of the boundary of a closed subset into areas from where some or all solutions to the differential inclusion remain or leave this closed subset.

1. Lyapunov functions. Let $F : X = \mathbb{R}^n \rightsquigarrow X$ be a set-valued map with which we associate the differential inclusion

$$\text{for almost all } t \geq 0, x'(t) \in F(x(t)) \quad (1)$$

We also consider a time-dependent function $w(\cdot)$ defined as solutions to a differential equation

$$w'(t) = -\phi(w(t)), \quad (2)$$

where $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a given continuous function with linear growth.

We consider a nonnegative function $V : X \mapsto \mathbb{R}_+ \cup \{+\infty\}$ (called *extended function*), proper in the sense that its domain

$$\text{Dom}(V) := \{x \in X \mid V(x) < +\infty\} \quad (3)$$

is not empty. We assume that $\text{Dom}(V) \subset \text{Dom}(F)$. This function V is said to enjoy the ϕ -Lyapunov property, i.e.,

$$\forall t \geq 0, V(x(t)) \leq w(t), \quad w(0) = V(x(0)) \quad (4)$$

Received July 19, 1988.

AMS Subject Classifications: 34A60, 34D20, 49C15.