Differential and Integral Equations, Volume 1, Number 1, January 1988, pp. 1–21.

GENERALIZATION OF THE SOBOLEV-LIEB-THIRRING INEQUALITIES AND APPLICATIONS TO THE DIMENSION OF ATTRACTORS

JEAN-MICHEL GHIDAGLIA, MARTINE MARION, AND ROGER TEMAM
Laboratoire d'Analyse Numérique, C.N.R.S. et Université Paris-Sud
Bâtiment 425, Université Paris-Sud, 91405 Orsay Cédex, France

Abstract. In relation to the question of stability of matter, Lieb and Thirring have given a remarkable improvement of Sobolev-Gagliardo- Nirenberg inequalities for finite families of functions which are orthonormal in $L^2(\mathbf{R}^n)$. These inequalities have also found utilization in the study of the dimension of the attractors associated with dissipative parabolic equations, in particular Navier-Stokes equations in a bounded domain with zero boundary conditions, where they have led to sharp estimates on this dimension. The generalizations that we propose here are partly motivated by the study of other equations of fluid mechanics for which the previous inequalities do not apply. Our extensions are made possible, in particular, by the fact that we replace the global condition of orthonormality in $L^2(\mathbf{R}^n)$ by a local one which is called suborthonormality condition (this concept is defined hereafter). We are hence able to consider arbitrary boundary conditions; we can also consider higher order derivatives and vector valued functions. The last Section of the paper contains some applications to viscous incompressible fluid flows which illustrate some of the generalized inequalities.

1. Introduction. In relation to the question of stability of matter, Lieb and Thirring [19] have proved a remarkable improvement of Sobolev- Gagliardo-Nirenberg inequalities for a finite family of functions which are orthonormal in $L^2(\mathbf{R}^n)$.

Let $\varphi_1, \dots, \varphi_N$ be a finite family of functions in $H^1(\mathbf{R}^n)$ which are orthonormal in $L^2(\mathbf{R}^n)$ i.e.

$$\int_{\mathbf{R}^n} \varphi_i \varphi_j \, dx = \delta_{ij}, \qquad 1 \leq i, j \leq N. \tag{1.1}$$

According to the extension of the Sobolev-Gagliardo-Nirenberg inequalities due to Lieb and Thirring [19] (see also Cwickel [6]), for every p satisfying $\max(1, n/2) < p \leq 1 + (n/2)$, there exists a constant $\kappa = \kappa(n,p)$ independent of N and of the φ_j's such that

$$\Big[\int_{\mathbf{R}^n} \Big(\sum_{j=1}^N \varphi_j(x)^2\Big)^{p/(p-1)} dx\Big]^{2(p-1)/n} \leq \kappa \sum_{j=1}^N \int_{\mathbf{R}^n} \sum_{i=1}^n \Big(\frac{\partial \varphi_j}{\partial x_i}\Big)^2 dx. \tag{1.2}$$

Besides its initial motivation for the stability of matter, this inequality has played an important role in the estimate of the trace of certain linear operators arising in the study of

Received June 23, 1987.
AMS(MOS) Subject Classifications: 26D10, 35Q10, 35Q20.