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# SINGULAR SOLUTIONS OF BESSEL'S EQUATION

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**Abstract.** We show that for the Bessel equation of integer order $n$, there is a singular solution consisting of a finite sum of Dirac delta function's derivatives. It is intimately related to the Tchebycheff polynomial of the second kind of degree $n-1$.

**I. Introduction.** Distribution theory has been used to solve ordinary differential equations concerning weight functions of orthogonal polynomials for approximately ten years. In one formulation the weights are given as an infinite sum of Dirac delta function derivatives. By using the Fourier transform, the classical weight functions can be recovered from these series [4].

More recently [3], the idea of using a series of delta function derivatives has been applied to such classical problems as the hypergeometric equation, the confluent hypergeometric equation and the general Bessel equation. Traditional application of the Laplace transform to solve Bessel's equation of order 0 has been known for years. Both $J_0$ and $Y_0$ have Laplace transforms. Application of the Laplace transform in the traditional sense to Bessel's equation of order $n$ yields only $J_n$, since $Y_n$ is too singular at 0 to have a Laplace transform. Distributionally, however, the Laplace transform yields two solutions to Bessel's equation. In general, both are infinite delta function derivative series, one of which can be identified with $J_n$. When $n$ is an integer, one or the other of these series terminates [3].

The purpose of this article is to derive quickly and concisely via the Fourier transform these finite series of derivatives of the Dirac delta function, which are singular solutions of the Bessel's equation of integer order $n$. There is a second solution which we do not discuss. It can be identified with $J_n$, [3].

**II. Singular solutions of Bessel's equation.** We denote by $S$ the space of test functions of rapid decay [2]. The space of continuous linear functionals on $S$, distributions of slow growth, is denoted by $S'$. It is well known that every element in $S$ has a Fourier transform which is also in $S$. As a consequence, every element in $S'$ has a distributional Fourier transform, which is also in $S'$. We choose $S'$ as our setting.

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