AUTOMORPHISMS AND BACKLUND TRANSFORMATIONS

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Abstract. Bäcklund transformations for completely integrable systems are obtained as gauge transformations of the connection associated with the system. These gauge transformations depend projectively on the wave functions, and the differential equation for the gauge transformation leads in a natural way to the Riccati equations for the inhomogeneous coordinates of the wave functions. In some cases, such as the AKNS system, the Bäcklund transformation is associated with a simple automorphism of the underlying Lie algebra. The Bäcklund/gauge transformation for the modified Lax equations is constructed, and the associated automorphism found by Fordy is discussed. An automorphism for Bäcklund transformations for the generalized wave equation, discussed by Beals and Tenenblat, is given.

1. Introduction. There has been considerable interest in Bäcklund transformations for completely integrable systems in recent years, and the subject has been approached from a number of different directions. In one point of view, completely integrable systems are viewed as integrability conditions for a flat connection, and the Bäcklund transformation is obtained via a gauge transformation that intertwines with the components of the connection. The gauge-theoretic point of view has been pursued by numerous researchers (cf. [4], [5], [8], [10], [11], [13], [14]). It is the analog of the Darboux transformation in the theory of the Schrödinger equation and Bäcklund transformations for the KdV equation.

In another point of view, Riccati equations for the ratios of the components of the wave functions are derived from the linear equations for the wave functions, and the Bäcklund transformations are obtained from suitable automorphisms of these Riccati equations. This approach has been developed by Chen [6], Fordy [7], Winternitz [18], and Harnad, St. Aubin, and Shnider [9], among others.

In fact, the gauge transformations depend projectively on the wave functions, in a sense we shall explain here, and are found to satisfy a matrix Riccati equation. When rewritten in terms of the wave functions, the equations satisfied by the gauge transformation lead naturally to the Riccati equations for the wave functions as well as the automorphism associated with the Bäcklund transformation. These facts follow quite simply from the results in [14], and will be explained below.