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SOME EXTENSIONS OF THE MOUNTAIN PASS LEMMA

DAJUN GUO, JINGXIAN SUN AND GUIJIE QI
Department of Mathematics, Shandong University
Jinan, Shandong, P.R. of China

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0. Introduction. In 1973, A. Ambrosetti and P.H. Rabinowitz [1] proved the well-known Mountain Pass Lemma as follows:

Lemma 1. (Mountain Pass Lemma). *Let E be a real Banach space, $f : E \to R^1$ be a C^1 functional satisfying the (P.S.) condition, $x_0 \in E$, $x_1 \in E$, D be an open neighborhood of x_0; $x_1 \bar{\in} \bar{D}$. Suppose that*

$$\inf_{x \in \partial D} f(x) > \max\{f(x_0), f(x_1)\}. \tag{1}$$

Let

$$c = \inf_{h \in \Phi} \max_{t \in [0,1]} f(h(t)), \tag{2}$$

where $\Phi = \{h \,|\, h : [0,1] \to E$ is continuous, $h(0) = x_0$, $h(1) = x_1\}$. Then c is a critical value of f.

Some authors (for example, P. Pucci and J. Serrin, and others) attempted to improve condition (1). In [2], P. Pucci and J. Serrin proved that if condition (1) is replaced by one as follows:

$$\inf_{x \in N_r} f(x) \geq \max\{f(x_0), f(x_1)\}, \tag{3}$$

where $N_r = \{x \in E \,|\, \operatorname{dist}(x, \partial D) < r\}$, $(r > 0)$, then c defined by (2) is a critical value of f. In [3], P.H. Rabinowitz points out that if condition (1) is replaced by

$$\inf_{x \in \partial D} f(x) \geq \max\{f(x_0), f(x_1)\}, \tag{4}$$

then $f(x)$ has a critical value, however, defined by a different minimax procedure. Whether Lemma 1 holds with the condition (1) replaced by (4) is an open question. In this paper one of our major objectives is to give a positive answer to this question. In fact, we prove that if condition (1) is replaced by (4), then $K_c \backslash \{x_0, x_1\} \neq \emptyset$, where $K_c = \{x \in E \,|\, f(x) = c,\ f'(x) = \theta\}$, and c is defined by (2). In addition, we also give some conclusions relating saddle point or mountain-pass type under condition (4). In §2 we will consider the Mountain Pass Lemma on closed convex set in Hilbert space.

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