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ON THE INVERSION OF LAGRANGE-DIRICHLET THEOREM*

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Abstract. The inversion of the Lagrange-Dirichlet theorem is proved under the hypothesis that the potential function U of the acting force is h-differentiable, $h > 3$, and the lack of a local maximum of U at the equilibrium position is recognizable by means of the non-vanishing terms with lowest degree in the expansion of U. This result extends a previous one relative to infinitely differentiable potential functions and is obtained by using known results concerning the existence of invariant stable manifolds.

Introduction. The Lagrange-Dirichlet theorem, as is well known, provides a sufficient condition for the stability of an equilibrium position of a conservative mechanical system. Precisely, let S be a holonomic mechanical system with a finite number n of degrees of freedom and let $q = (q_1, \cdots, q_n)$ be a system of Lagrangian coordinates for S. Let us suppose that a conservative force with potential function $U : \Omega \to \mathbb{R}$, Ω neighborhood of the origin of $\mathbb{R}^n$, $U \in C^h$, $h \geq 2$, acts on S. Finally, let $q = 0$ be an equilibrium position of S. The L.-D. theorem assures that $q = 0$ is stable if U has a strict local maximum at $q = 0$. As also is well known, the L.-D. criterium is not invertible. Therefore, the question arises: under what additional conditions the lack of a strict local maximum of U at $q = 0$ implies the instability of this equilibrium position. Starting from Liapunov, many answers have been given. We will quote some of the most relevant ones. Denoting by $U_{[i]}$, $i = 2, \cdots, h$, the term of degree i in the development of U in the neighborhood of the origin, the following criteria of instability hold. The equilibrium position $q = 0$ is unstable if one of the following conditions holds:

- i_1) $U_{[2]}$ does not have a maximum at $q = 0$ (Liapunov [7]);
- i_2) $h > 2$, $\exists$ a positive integer k, $2 < k \leq h$, such that $U_{[2]} = \cdots = U_{[k-1]} = 0$ and $U_{[k]}$ has a proper minimum at $q = 0$ (Liapunov [7]);
- i_3) U is an homogeneous polynomial and does not have a maximum at $q = 0$ (Cetaev [1]);
- i_4) U has a proper local minimum at $q = 0$ (Hagedorn [3]);
- i_5) $h > 2$, $\exists$ a positive integer k, $2 < k \leq h$, such that $U_{[2]} = \cdots = U_{[k-1]} = 0$, $q = 0$ is an isolated critical point for $U_{[k]}$, and $U_{[k]}$ does not have a maximum at $q = 0$ (Palamadov [8]);

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