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EXISTENCE AND REGULARITY FOR A SINGULAR SEMILINEAR STURM-LIOUVILLE PROBLEM

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Abstract. In this paper, we consider the following Sturm-Liouville problem

$$-\frac{1}{r^\gamma}(r^\gamma u')' = r^\beta |u|^{p-1}u \quad \text{in } (0,R)$$
$$u(R) = 0, \qquad \int_0^R r^\gamma |u'|^2\, dr < \infty$$

where $p \geq 1$, $\beta > -2$, $\gamma \in \mathbb{R}$. Using the Mountain Pass Lemma we prove the existence of a weak positive solution under optimal conditions on the parameters. Beyond these conditions a variant of Pohozaev's identity gives non-existence. Using Möser's iteration technique we prove the boundedness of this solution which in turn gives the uniqueness. Applying a symmetric version of the Mountain Pass Lemma we proved the existence of infinitely many weak solutions changing sign. The main tool in the proof is the generalized Hardy-Litlewood inequality. We apply these results to semilinear degenerate elliptic equations in $\mathbb{R}^N$ and we get new interesting corollaries.

0. Introduction. We consider the following boundary value problem

$$\begin{aligned} -\text{div}\,(r^\alpha \nabla u) &= r^\delta |u|^{p-1}u \quad \text{in } B_R(0) \subset \mathbb{R}^N,\ N \geq 3,\ r = |x| \\ u &= 0 \quad \text{on } |x| = R. \end{aligned} \tag{0.1}$$

If we restrict our search to radially symmetric solutions then this equation reduces to the following one

$$\begin{aligned} -\frac{1}{r^\gamma}(r^\gamma u')' &= r^\beta u^p \quad \text{in}\,(0,R),\ \gamma = N + \alpha - 1,\ \beta = \delta - \alpha \\ u(R) &= 0 + \text{a boundedness condition at } r = 0. \end{aligned} \tag{0.2}$$

One is tempted to set $u'(0) = 0$ but this is too restrictive since as we shall see later there exist weak solutions of (0.2) which are bounded at 0 and are not differentiable at this point. The natural boundedness condition is

$$\int_0^R r^\gamma |u'|^2\, dr < \infty \tag{0.3}$$

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