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ON NONUNIFORM ASYMPTOTIC STABILITY FOR NONAUTONOMOUS FUNCTIONAL DIFFERENTIAL EQUATIONS

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1. Introduction. Denote by C the space of continuous functions $\varphi : [-h, 0] \to \mathbb{R}^n$. For $\varphi \in C$ we will use the norms $\|\varphi\| := \max_{-h\leq s\leq 0} |\varphi(s)|$ and $|||\varphi||| := [\int_{-h}^{0} \varphi^2(s)\,ds]^{1/2}$. Given $H > 0$, C_H denotes the set of $\varphi \in C$ with $\|\varphi\| < H$. If $x : [t_0 - h, T) \to \mathbb{R}^n$ $(0 \leq t_0 < T \leq \infty)$ is continuous and $t \in [t_0, T)$, we define $x_t \in C$ by $x_t(s) = x(t+s)$ for $s \in [-h, 0]$. Let $x'(t)$ denote the right-hand derivative at t if it exists and is finite.

Consider the system

$$x'(t) = F(t, x_t) \tag{1.1}$$

where $F : \mathbb{R}_+ \times C_H \to \mathbb{R}^n$ is continuous and takes bounded set into bounded sets, $\mathbb{R}_+ := [0, \infty)$, $0 < H \leq \infty$. It is known [1] that for each $t_o \in \mathbb{R}_+$ and each $\varphi \in C_H$ there is at least one solution $x(t_0, \varphi) = x(\cdot, t_0, \varphi)$ defined on an interval $[t_0, t_0 + \alpha)$, $\alpha > 0$. We suppose that $F(t, 0) \equiv 0$ so that $x = 0$ is a solution of (1.1) and is called the zero solution.

Generalizing Lyapunov's classical stability theory on ordinary differential equations (ODE's) to functional differential equations (FDE's), Krasovskii [2] replaced the Lyapunov function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ with a continuous functional $V : \mathbb{R}_+ \times C_H \to \mathbb{R}$, whose derivative V' with respect to (1.1) was defined by

$$V'(t, \varphi) := \limsup_{\delta\to 0}[V(t + \delta, x_{t+\delta}(t, \varphi)) - V(t, \varphi)]/\delta.$$

Definition 1.1. *The zero solution of (1.1) is said to be stable if for each $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$ there is a $\delta = \delta(\epsilon, t_0) > 0$ such that $[\varphi \in C_\delta,\ t \geq t_0]$ imply that $|x(t, t_0, \varphi)| < \epsilon$. If δ is independent of t_0, then the zero solution is uniformly stable (U.S.). The zero solution is asymptotically stable (A.S.) if it is stable and if for each $t_0 \in \mathbb{R}_+$ there is a $\sigma = \sigma(t_0) > 0$ such that $\varphi \in C_\sigma$ implies that $x(t, t_0, \varphi) \to 0$ as $t \to \infty$. The zero solution is uniformly asymptotically stable (U.A.S.) if it is U.S.*

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