Differential and Integral Equations, Volume 3, Number 2, March 1990, pp. 201–207.

STABILITY AND UNIQUENESS OF POSITIVE SOLUTIONS FOR A SEMI-LINEAR ELLIPTIC BOUNDARY VALUE PROBLEM

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Abstract. Positive solutions of the semilinear elliptic boundary value problem $-\Delta u(x) = g(x)f(u(x))$ on D, $Bu = 0$ on ∂D are studied where D is a bounded region and f is concave. It is proved that every positive non-constant solution is linearly stable and using fixed point index arguments results on existence and uniqueness of positive solutions are deduced. The results obtained are well-known in the case where g is positive on D; the proof presented in this paper applies also to the case, arising in population genetics, when g changes sign on D.

In this paper, we consider the stability, existence and uniqueness of positive solutions of the semi-linear elliptic boundary value problem

$$-\Delta u(x) = g(x)f(u(x)) \quad \text{in } D; \qquad B_\alpha u(x) = 0 \quad \text{on } \partial D \tag{1}$$

where D is a bounded region in $\mathbb{R}^n$ with smooth boundary, $g : D \to \mathbb{R}$ is a smooth function, and $B_\alpha u(x) = \alpha h(x)u(x) + (1-\alpha)\frac{\partial u}{\partial n}$ where $\alpha \in [0,1]$ is a constant and $h : \partial D \to \mathbb{R}^+$ is a smooth function with $h \equiv 1$ when $\alpha = 1$, i.e., the boundary condition may be of Dirichlet, Neumann or mixed type. We shall assume throughout that f satisfies

(f1) $f : I \to \mathbb{R}^+$ is a smooth function where $I = [0, r]$ or $[0, \infty)$, $f(0) = 0$ and $f''(u) < 0$ for all $u \in I$.

We say that u is a positive solution of (1) if u is a classical solution with $u(x) \in I$ for all $x \in \overline{D}$ and $u(x) > 0$ for all $x \in D$.

Our study of (1) is motivated by the fact that the equation arises in population genetics (see [5]) in which case the function g attains both positive and negative values on D. In the case when $g \equiv 1$ it is well known that (1) has at most one non-constant positive solution when f satisfies (f1) (see e.g., Cohen and Laetsch [4]) but may have multiple solutions when f is convex (see e.g. Amann [1]). At first sight it seems unlikely that a uniqueness theorem should hold for (1) under the assumption (f1) when g changes sign on D as $u \to g(x)f(u)$ is convex whenever

Received May 22, 1989.
AMS Subject Classifications: 35J65, 35P30.