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ON DIFFERENTIAL EQUATIONS IN ORDERED BANACH SPACES WITH APPLICATIONS TO DIFFERENTIAL SYSTEMS AND RANDOM EQUATIONS

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1. Introduction. Given a closed cone K in a Banach space E, consider the initial value problem (IVP)

$$x'(t) = f(t, x(t)) \text{ a.e. } t \in I, \quad x(0) = c, \tag{1}$$

where $I = [0, T]$, $T > 0$, $c \in K$ and $f : I \times K \to E$. By a solution of (1) we mean a continuous function $x\colon I \to K$ which is almost everywhere differentiable on I, for which $\|x(\cdot)\|$ is absolutely continuous, and which satisfies (1) (cf. [14]). Thus, x is a solution of (1) if and only if it satisfies the integral equation

$$x(t) = c + \int_0^t f(s, x(s))\, ds \tag{2}$$

on I. An existence of such a solution x necessitates only Bochner integrability of $f(\cdot, x(\cdot))$ on I, which allows f to be rather discontinuous (cf. [13]). But this is not sufficient. If, for instance, $E = \mathbf{R}$, $0 < \delta \le T$, and

$$f(t, y) = \begin{cases} 1, & \text{for } y = 0,\ \ t \in [0, \delta], \\ 0, & \text{for other } (t, y) \in I \times \mathbf{R}_+ , \end{cases}$$

it is easy to see that IVP (1) is nonsolvable, even locally, as $c = 0$. However, f is continuous on $I \times \mathbf{R}_+$, except on the segment $[0, \delta] \times \{0\}$.

The classical Carathéodory conditions (cf. [2]), which ensure the solvability of (1) for each choice of c when $E = \mathbf{R}^n$, do not allow the previous kind of discontinuity. In fact, they imply (cf. [10]) for each $\delta > 0$ the existence of a closed subset B of I whose L-measure is greater than $T - \delta$, such that f is jointly continuous on $B \times \mathbf{R}^n$.

If E is infinite-dimensional, the Carathéodory conditions do not ensure the solvability of IVP (1). Counter examples, where f is even continuous, are found for instance in [3], where is also given sufficient strengthenings to Carathéodory conditions (uniform continuity of $f(t, y)$ in y, etc; see also [11]).

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