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NONLINEAR SCHROEDINGER EQUATIONS WITH MAGNETIC FIELDS

ANNE DE BOUARD
Laboratoire d'Analyse Numérique, CNRS et Université Paris-Sud
Bâtiment 425, 91405 Orsay, France

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Abstract. We study the Cauchy problem for nonlinear Schrödinger equations with magnetic field. Under some growth conditions on the potentials, we show the existence of solutions in $L^2(\mathbb{R}^n)$ and in a weighted Sobolev space Σ. We also establish the continuous dependence on the initial value, and the conservation of energy when the solution is in Σ.

1. Introduction. We consider the nonlinear Schrödinger equation in $\mathbb{R}^n$:

$$i\frac{\partial u}{\partial t} = \frac{1}{2}\sum_{j=1}^{n}\left(-i\partial_j - A_j(x)\right)^2 u + V(x)u + \epsilon|u|^{p-1}u, \tag{1.1}$$

where $u = u(t,x)$ is a complex-valued function defined on $[-T,T] \times \mathbb{R}^n$ for some $T > 0$. We show the local existence of solutions for the Cauchy problem in the function space

$$\Sigma = \left\{u \in S'(\mathbb{R}^n),\ (1+|x|^2)^{1/2}u \in L^2(\mathbb{R}^n),\ (I-\Delta)^{1/2}u \in L^2(\mathbb{R}^n)\right\}$$

if $1 \le p < 1 + \frac{4}{n-2}$, and in $L^2(\mathbb{R}^n)$ if $1 \le p \le 1 + \frac{4}{n}$, when the vector potential $A = (A_1(x), \dots, A_n(x))$ and the scalar potential V are smooth and satisfy the same growth conditions as in [15] (see below for a precise statement). We choose this power nonlinearity for simplicity, but we can allow more general terms, as in [7] or [10].

We denote by H the operator associated with the steady linear equation. Such operators have been studied for example in [4], [12] or [15].

In all the paper, our assumptions will be the following:

H1: We assume that for $j \in \{1,\dots,n\}$, $A_j(x)$ is real valued, C^∞ on $\mathbb{R}^n$. If $B = (B_{jk})$ with $B_{jk} = \partial_j A_k - \partial_k A_j$, then there exists $\epsilon > 0$ such that

$$|\partial^\alpha B(x)| \le C_\alpha(1+|x|)^{-1-\epsilon}, \quad \forall|\alpha| \ge 1,\ \forall x \in \mathbb{R}^n,$$

$$|\partial^\alpha A(x)| \le C_\alpha, \forall|\alpha| \ge 1,\ \forall x \in \mathbb{R}^n.$$

H2: V is real valued, C^∞ on $\mathbb{R}^n$, $|\partial^\alpha V(x)| \le C_\alpha$, $\forall|\alpha| \ge 2$; in addition we assume that V is bounded from below; i.e., we can assume that there exists $m > 0$ such that $V(x) \ge m$, $\forall x \in \mathbb{R}^n$.

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