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ON THE DIMENSION OF ATTRACTORS FOR NAVIER-STOKES EQUATIONS ON TWO-DIMENSIONAL COMPACT MANIFOLDS

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Abstract. We study Navier-Stokes equations on a two-dimensional compact manifold M,

$$\partial_t u + \nabla_u u - \nu\Delta u = -\nabla p + f, \quad \text{div } u = 0,$$

in the phase space of divergence free vector fields on M including the n-dimensional space of harmonic vector fields. We derive upper bounds for the Hausdorff and fractal dimensions of the global attractor which are logarithmically close to being optimal. For the most important case $M = S^2$ (the unit sphere), the explicit values of constants are given and possible applications to large-scale atmospheric dynamics are discussed.

1. Introduction. Navier-Stokes and Euler equations arise quite naturally from different problems of meteorology. For instance, the vorticity form of Navier-Stokes equations on the β-plane or on the rotating sphere is widely used in dynamical meteorology ([5], [6], [9], [14], [15], [19], [21], [28]).

The equations of hydrodynamics on manifolds have been studied in [7], [10], [11], [23], with $\Delta u = (\Delta u)^i = g^{kl}\nabla_k\nabla_l u^i$ standing for the viscous dissipation term. In [13], we proposed a different form of dissipation and used the Laplace-de Rham operator $\Delta = -d\delta - \delta d$. Of course, for the flat case ($g_{ij} = \delta_{ij}$) these two operators coincide, but in the presence of curvature the Laplace-de Rham operator has some additional useful properties. First, the vorticity equation has the same form as that in the flat case; second, we have the same orthogonality relation as in the space periodic problem.

For the Navier-Stokes equations in the bounded domain with classical non-slip boundary condition, R. Temam [23] obtained the following bound for the Hausdorff and fractal dimensions of the attractor: $\dim \mathcal{A} \le cG$, $G = \frac{\|f\|}{\lambda_1\nu^2}$ is the generalized Grashof number [8]. This estimate has been improved by P. Constantin, C. Foias, and R. Temam ([3], [23]) for Navier-Stokes equations with periodic boundary conditions corresponding to $M = T^2$ with flat metric $g_{ij} = \delta_{ij}$; $\dim \mathcal{A} \le cG^{2/3}(1 + \ln G)^{1/3}$. In [11], the Sobolev-Lieb-Thirring inequalities playing an essential role in the study of the dimension of the attractors have been extended

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