Differential and Integral Equations, Volume 6, Number 1, January 1993, pp. 37–45.

PSEUDO-MONOTONICITY AND THE LERAY-LIONS CONDITION

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(Submitted by: Peter Hess)

1. Introduction. Let Ω be a bounded open subset of $\mathbb{R}^N$, let $1 < p < \infty$ and let T be the mapping from $W_0^{1,p}(\Omega)$ to $W^{-1,p'}(\Omega)$ which is generated by the differential operator

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, u, \nabla u) + a_0(x, u, \nabla u), \tag{1.1}$$

where the functions $a_i(x, \eta, \zeta)$ and $a_0(x, \eta, \zeta)$ satisfy suitable regularity and growth assumptions. The standard condition which guarantees that T is pseudo-monotone is the so-called Leray-Lions condition:

$$\sum_{i=1}^{N} (a_i(x, \eta, \zeta) - a_i(x, \eta, \bar{\zeta}))(\zeta_i - \bar{\zeta}_i) > 0 \tag{1.2}$$

for almost every $x \in \Omega$, all $\eta \in \mathbb{R}$ and all ζ, $\bar{\zeta} \in \mathbb{R}^N$ with $\zeta \neq \bar{\zeta}$ (cf. [2, 7, 8]). In this paper we are interested in the necessity of condition (1.2) as well as in the sufficiency of a related weaker condition.

It is known that the pseudo-monotonicity of T implies that

$$\sum_{i=1}^{N} (a_i(x, \eta, \zeta) - a_i(x, \eta, \bar{\zeta}))(\zeta_i - \bar{\zeta}_i) \geq 0 \tag{1.3}$$

for almost every $x \in \Omega$, all $\eta \in \mathbb{R}$ and all ζ, $\bar{\zeta} \in \mathbb{R}^N$. This is proved in [1] when $a_0 \equiv 0$, but the arguments given there easily extend to the situation $a_0 \not\equiv 0$. In Section 2 we study the necessity of the *strict* condition (1.2). We restrict ourselves there to operators where the top order terms a_i only depend on ∇u. We show that if, for any lower order term a_0, the corresponding mapping T is pseudo-monotone, then the top order terms a_i satisfy (1.2). Condition (1.2) thus appears, for this class

Received December 1991.
AMS Subject Classification: 35J65, 47H05.