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APPROXIMATE INERTIAL MANIFOLDS FOR THE SINE-GORDON EQUATION

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1. Introduction. The subject of this article is to study the long time behaviour of the well-known Sine-Gordon equation, which has a particular interest in physics. The case we consider here is

$$\begin{cases} u'' + \alpha u' - \Delta u + j \sin u = 0 \\ u(x,t) = 0 \quad \text{on } \partial\Omega \\ u(x,0) = u_0, \quad u'(x,0) = u_1 \\ \Omega \subset \mathbb{R}^n, \quad n \leq 3. \end{cases}$$

This problem has already been studied and we know it possesses a global attractor of finite Hausdorff dimension ([3]). The existence and non existence of an inertial manifold, which is a finite dimensional Lipschitz invariant manifold attracting exponentially all the orbits when t goes to infinity, has been proved ([10]) in dimension $n = 1$.

But even when their existence is known, the computation of these manifolds is still very difficult. That is why the concept of Approximate Inertial Manifolds ([5]) has been introduced. It is a finite dimensional smooth manifold such that all the orbits enter after a transient time a very thin neighborhood of the manifold. The existence of AIM's has been shown, for instance, for the 2D Navier-Stokes equation ([5], [12]), for reaction diffusion equations ([7]) and for the Cahn-Hilliard equation ([8]). An infinite family of AIM's has even been constructed for these problems ([2], [12]). These manifolds are specially interesting because they lead to new numerical schemes, well adapted to the long term integration of evolution equations ([9], [11]). Our aim in this paper is to construct an infinite family of AIM's for the Sine-Gordon equation, providing better and better order approximations to the orbits. We thus consider the orthonormal basis of $H = L^2(\Omega)$ consisting of the eigenvectors of the Laplacian

$$\Delta w_j = \lambda_j w_j, \quad j = 1, 2, \ldots, \quad 0 < \lambda_1 \leq \lambda_2 \leq \ldots, \quad \lambda_j \to +\infty \text{ as } j \to +\infty.$$

For a fixed m, we denote $P = P_m$ the projector in H onto $Sp(w_1, \ldots, w_m)$, and $Q = Q_m = I - P_m$.

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